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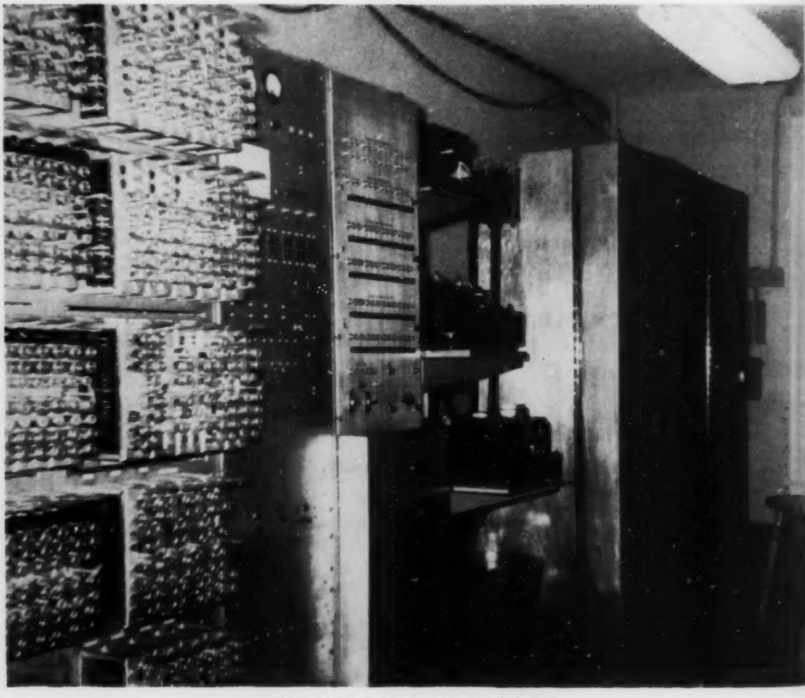

THE ONR RELAY COMPUTER

The Office of Naval Research Relay Computer

1. **Introduction.** This paper describes a general purpose relay computer. This digital machine was installed in Staughton Hall on the campus of The George Washington University on May 9, 1951. At that time it was renamed the "ONR Relay Computer" and was made available by the Office of Naval Research to the Logistics Research Project for operation and maintenance under an ONR–GWU contract. The machine is a slow-speed, single-address computer with magnetic drum storage of 4094 numbers or instructions, each consisting of 24 binary digits including sign (equivalent to about seven decimal digits). It is capable of performing 42 arithmetic, logical, transfer, output and stop instructions through use of 734 mechanical relays and 655 electronic tubes. Input to the computer is on mechanically sensed, seven-level teletype paper tape, while output may be on electric typewriter, or on punched tape. A view of the computer is shown in the frontispiece.

2. **Storage.** The storage element is a cylinder or drum 12 inches in diameter, $8\frac{1}{2}$ inches long. The cylindrical surface of this drum is coated with a ferromagnetic material, small areas of which may be magnetized in either of two directions to represent the binary digits 0 or 1. The drum rotates at 440 revolutions per minute, and the magnetization configurations can be played back from or recorded (stored) on it 24 at a time by a group of 24 magnetic heads placed .002 inches from the surface. Two additional playback heads are required for timing. All electronic tubes in the computer are used in connection with the storage unit; they accomplish amplification, shaping and timing of signals in the memory unit. These groups of 24 binary digits represent both numbers and coded instructions to the computer, either of which may be stored on any part of the drum. Successive memory locations are normally loaded from punched paper tape sensed by a standard mechanical tape reader, but any can be loaded singly by manual controls. Each group of 24 binary digits has an address of 12 binary digits which refers to and specifies its particular location on the drum. There are 4094 such addresses, from octal 0002 through 7777. Given any address, the operator can inspect by manual control the contents of the address in a display of neon lights.

3. **Arithmetic Unit.** All arithmetic is performed by mechanical relays within the computer. A relay may be either open (nonconducting) or closed (conducting), representing the binary digits 0 and 1, respectively. The computer contains three basic arithmetic registers which are groups of relays: the Accumulator, the Q register and the X register. The Accumulator, A, is capable of binary representation of a number of 48 digits. It is the principal arithmetic register, the place where the actual arithmetic is performed, and possesses subtracting and shifting properties. The 24-binary-digit Q register holds the multiplier during multiplication and the quotient after division. The 24-binary-digit X register serves as a temporary storage for both numbers and commands as they come from the memory. In it are

held instructions during decoding and data while being operated upon. For example, the X register contains the multiplicand during multiplication.

The binary number system is used throughout. On the control panel lights and buttons are grouped in triplets for easier octal representation. All numbers are considered as integers by the computer, with the left-most digit indicating the sign (0 designates positive; 1, negative). A negative number is represented by the "one's" complement of the corresponding positive number. The Accumulator is basically a subtractor, so additions are actually performed by complementing one of the numbers to be added and subtracting it with end-around borrow from the other number. Such subtractions are performed in parallel; i.e., all 24 digits are operated on at once. Multiplication is a series of subtractions and shift operations.

Conceptually, the arithmetic unit is much like a desk calculator where the Accumulator corresponds to the product or upper dials, with twice as many digits as the keyboard; the Q register corresponds to the lower or multiplier dials, equal in length to the keyboard; and the X register is similar to the keyboard itself. But the Relay Computer's "keyboard" X can effectively be filled from any of the 4094 addresses under control of instructions on the drum. Basically the arithmetic unit is like an automatically sequenced desk calculator with 4094 keyboards.

4. Instructions. Each instruction occupies one address. Such an instruction always contains an operation code for the left-hand six binary digits. These two octal digits designate which of the 42 instructions the computer is to perform. The right-hand four octal digits usually indicate the address from which data are to be taken for use with the instruction being performed. Some instructions do not need an associated address; they use the allotted digits for other purposes. In the shift instruction, for example, these digits indicate the number of digit-positions to be shifted. In any case the third and fourth octal digits from the left are not interpreted by the computer in the execution of an instruction.

While the computer is in operation, the arithmetic section operates asynchronously with the memory section, each waiting in turn for the other to perform its function. In normal operation, when the address of the first instruction has been sent to the memory unit (to be held in the Drum Address Register, DAR), further action is suspended until that address has passed under the playback heads and its contents have been transferred to the X register. While the 6-binary-digit operation code is being decoded, the 12-binary-digit address, if contained in the instruction, is transferred back to the Drum Address Register. No arithmetic operation can be performed until the specified address has been located on the drum and its contents have been transferred to the X register. Actually an address is found electronically by circumferentially counting timing channel pulses and matching the count with the contents of the DAR. While the instruction is being executed, action in the memory unit is suspended. Then the address from which the next order is to be taken is transferred to the DAR. In this way the computer normally proceeds in linear sequence from one instruction to the next. The Instruction Address Register, IAR, holds the address of the next instruction in linear sequence to be sent to the DAR before the execution of each instruction. In normal sequence "one" is automatically added to the contents of the IAR as each instruction is carried out.

Four "jump" instructions provide important exceptions to the linear sequence described. Each one of these can specify the address from which the computer is to take its next instruction under a certain condition. That is, the four right-hand octal digits of these instructions themselves are transferred to the DAR under certain conditions; this transfer interrupts the normal transfer of the contents of the IAR to the DAR. These four octal digits are also transferred to the IAR, so that operation again proceeds in a linear sequence starting with the instruction at this new address. This ability to alter the linear sequence of instructions, together with the ability to alter instructions by performing simple arithmetic with them, is the basis for the real flexibility of this general purpose computer.

Following is a brief description of the instructions:

1) Fourteen insert instructions: These cause the transfer of 24 binary digits replacing the previous contents of a receiving location. Included are transfers between the drum, Accumulator and Q register. Besides these normal transfers, provision is made for the transfer of absolute values, complements and specified digits, and for the storage of the right-hand 15 digits (address part) from A.

2) Eleven arithmetic operations: They include special additions and subtractions to facilitate double precision (48-binary-digit) arithmetic and manipulations with absolute values, and summation of the contents of successive storage locations. In division the non-negative remainder is retained in A. Both clear multiply and accumulate multiply produce 48-binary-digit products in the Accumulator.

3) Four jump instructions: One is unconditional. Two others depend on the sign of A or Q (control is transferred if the register contains a negative number). The fourth is a zero test for the Accumulator.

4) Two shifting operations: These instructions provide for the circular left shifting of the contents of A or Q.

5) Three stops: optional, intermediate (which depend on pre-set switches), and final.

6) Three output instructions: One causes the typewriter to print a single character or perform a single function (carriage return, space, etc.). Another causes the typewriter to perform as above and also causes the punching of the corresponding teletype code on tape. The third causes the contents of successive storage locations to be punched in a form suitable for input or later printing.

7) Five logical operations: These include two digit-by-digit multiplication instructions, noncarry addition, clear half of A and a special "transfer and add one" instruction.

Division and multiplication (depending on the number of one's in the multiplier) require approximately $2\frac{1}{2}$ seconds. All other operations which refer to storage can be performed at the rate of 220 per minute, i.e., one every two drum revolutions or .272 seconds per single address operation. Those which do not require reference to the memory (shifts, jumps, transfers between A and Q) are executed with twice the speed. An exception to these rules is the zero-jump test of the Accumulator. This instruction, though reference to the memory is not necessary, requires the time of 2

drum revolutions since "one" is automatically subtracted from the contents of A after the zero test.

5. Input, Output, and Operation. Normal perforated input tapes contain seven channels across the tape, six of which are used for binary information. Four lines, therefore, contain 24 binary digits of information—a complete number or instruction. Each line can be considered to have one octal digit on each side of the tape feed hole. The seventh channel normally contains a punch in every fourth line of the tape to signal the end of a 24-digit group. For normal input the operator inserts the tape in the reader and indicates the address into which the first group is to be stored by pressing digit keys of the Drum Address Register. When reading has started, successive groups on tape are automatically stored at successive addresses as long as holes continue to appear in the seventh channel in every fourth line. This normal tape input takes about ⅔ second for each 24 binary digits, i.e., 90 addresses are loaded per minute. Another type of drum loading from tape is possible. In this case each line (6 binary digits) of tape is loaded into the least significant six places of consecutive storage locations. This type of drum loading proceeds at 220 storage locations per minute.

Under manual control, not only can the operator inspect in lights the contents of any desired memory location, but he can cause insertion of 24 binary digits there. The contents of A, Q and X are always displayed in lights even during computation. Under manual control of the operator, as well as when program-controlled, the machine is capable of automatically punching the contents of successive storage locations onto tape in a form directly suitable for input when the first address is specified by the operator. Output of this kind is at the rate of 110 addresses per minute. Also, when operator- or program-controlled, the machine can sum the contents of consecutive storage locations. This operation, used for checking the storage of a complete program or a set of data whose sum is known, proceeds at the rate of 440 addresses per minute, leaving the cumulative sum in the Accumulator.

Actually the operator, when performing the manipulations discussed above, and the computer itself, while under control of the program, make use of three additional 24-binary-digit registers. The Storage Insertion Register contains a number to be stored in the memory unit; the Storage Output Register holds the number when it comes from the drum; and the Storage Blocking Register holds binary "1's" in digit positions where storage of a number is to be blocked. All three may be loaded manually by pushing buttons for binary 1's.

After both data and instructions are in the memory, the calculation may be started (there is no program instruction by which the computer can call for additional data to be stored on the drum during computation). When starting computation, the operator by means of a switch may cause the machine to perform a single instruction or to execute instructions successively from one to the next in normal operation. The computer stops automatically and indicates the trouble if the operation code it receives is a nonpermissible one, if it reaches a meaningless print code, or if in division the quotient is too large for the Q register.

6. Maintenance and Use. Ideally, after a 15 to 30 minute warm-up

period and a successful run of the test program each morning, the computer would be available for operation during the remainder of the day, since no routine maintenance periods are scheduled. The warm-up period may be prolonged or the operation interrupted for needed nonscheduled maintenance. Since no marginal checking features, computational checks or indications as to the faulty section are provided internally, much time is sometimes consumed in finding and remedying trouble. Ordinarily programs are written to include computational checks or are written so that a complete duplication of each computation section is performed before proceeding to the next.

The Logistics Research Project usually operates the computer 8½ hours a day during a five day week. For the first eleven months after delivery in May '51, the computer was maintained by several of The George Washington University undergraduate student engineers on duty a total of about 70 hours per week. This indicates that computer time was devoted to education of new maintenance men. Since then, one full-time engineer and one half-time undergraduate student have been maintaining the computer. During the first year of operation at the University, about $400 was spent for replacement parts and approximately $100 for parts for computer modification. This includes the cost of replacement of 21 electronic tubes and 50 relays. The maintenance men have undertaken projects other than routine maintenance. As an example, a test rack has been assembled which allows checking of any of the computer's 48 removable electronic chassis apart from the computer. During this first year of operation, approximately 42.5% of the time during which the machine was on was spent in maintenance, original checkout and education of maintenance men, and 2% for improvement or modification of the computer itself. Thus 55.5% of the time was available to the operator-programmers for the year. This last figure includes time spent in input, output, program checking and actual computation (which sometimes turns into time spent pinning down the cause of a machine error). For the first six months of operation the operator-programmer time was 43.4% of the total time; this increased to 68.3% for the other six months of the year. The computer, then, is purposely a research and educational tool in maintenance as well as operation.

Several problems from organizations outside the Logistics Research Project have been allotted machine time. These problems have been discussed in advance to determine eligibility for use of the machine with the Logistics Branch of the Office of Naval Research, Department of the Navy, or with the staff of the Logistics Research Project.[1] Tape preparation, duplication and print-out accessories are available to those whose problems have been accepted.

Following is a selected list of programs which were run during the first year, together with pertinent comments:

1) The Project used the machine to compute quarterly manpower requirements implied by a proposed four-year shipbuilding schedule. Given the building dates and total manpower requirement of a ship, a fourth-degree polynomial was evaluated to distribute the man-hour requirements into yearly quarters. These requirements were summed for the fleet.

2) A program was prepared for the iterative solution of games by

Brown's method—each player having 5 strategies. The method provides convergence to the value of the game within .01 after about 8 hours of computing (1000 iterations) for the games computed.

3) Solution of n simultaneous linear equations by Crout's method was performed for $n = 5$, and a general program was prepared for $n \leqslant 10$.

4) The Project used the computer to assist in obtaining a research solution to a fleet logistics planning problem.

5) Currently, a calculation is being performed which obtains information pertaining to feasible schedules for tanker delivery of fuel. Later this information will be used to obtain an optimal schedule with the hope of application to monthly delivery of petroleum products by the Military Sea Transport Service.

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This work was sponsored by the Office of Naval Research.

[1] Publications available from the Logistics Research Project are: Jos. B. Kruskal, Jr., *A Programmer's Description of* ABEL, *A Magnetic Drum Relay Computer, Orders for the* ONR *Relay Computer*, a detailed description of the instructions, and ONR *Relay Computer Operators Manual*.

On the Rounding Off of Difference Tables for Linear Interpolation

In order to simplify linear interpolation many tables contain the differences $\Delta(x) = f(x + d) - f(x)$ between two consecutive values of the function. Since the values of $f(x)$ given in the tables are usually rounded off the question arises whether the value of $\Delta(x)$ given in the table must be the difference between the rounded off values $\overline{f(x)}$ of $f(x)$ and $\overline{f(x + d)}$ of $f(x + d)$ or $\bar{\Delta}$, the result of the rounding off applied to $\Delta(x)$. It appears on the first view plausible that we obtain better results in the second case since we use here more information about $f(x)$. However, the detailed analysis shows that this is not so. If the values $f(x)$ are given with n decimals so that the rounding off errors do not exceed $h = \frac{1}{2}\,10^{-n}$, the part of the interpolation error due to the rounding off of $f(x)$ and $f(x + d)$ does not exceed h, if the difference used is $\overline{f(x + d)} - \overline{f(x)}$, while if we use $\bar{\Delta}$, this error can come arbitrarily near to $2h$.

Since this situation is not apparently realized by all computers of tables, I should like to develop an observation on this subject which was published elsewhere.[1]

We give first an example.

In computing the decimal logarithm log 9684.8 we start from the values $\log 9684 = 3.986\,054\,78$; $\log 9685 = 3.986\,099\,63$ and from the rounded off values $\log 9684 = 3.986\,05$; $\log 9685 = 3.986\,10$ with an error $< h = \frac{1}{2}\,10^{-5}$.

We have

$$\Delta = 4.485\cdot 10^{-5}, \qquad \bar{\Delta} = 4\cdot 10^{-5}.$$

We obtain then by the "complete" interpolation

$$\log 9684.8 = 3.986\,054\,78 + 0.8\cdot 4.485\cdot 10^{-5} = 3.986\,090\,66,$$

while in using the rounded off values of both logarithms and $\bar{\Delta}$ we obtain

$$3.986\,05 + 0.8 \cdot 4 \cdot 10^{-5} = 3.986\,082.$$

The difference is $.866 \cdot 10^{-5} > h$, while, if we take instead of $\bar{\Delta}$ the difference $5 \cdot 10^{-5}$ of the rounded off values, we obtain the value 3.986 09 and the whole rounding off error is $.066 \cdot 10^{-5} < h$.

From the following theoretical discussion it follows in particular, that if the rounding off errors of $f(x)$ and $f(x + d)$ can be considered as independent random variables, the probability that the rounding off error of the value of $f(x + td)$, obtained in using $\bar{\Delta}$, is $>(1 + \eta)h$, $0 \leqq \eta < 1$, is equal to $\frac{1}{4}(t - 2\eta + \eta^2/t)$, if $1 > t > \eta$. Without loss of generality, we can consider a function $f(x)$ with the two values $f(0) = f_0$, $f(1) = f_1$. Then for $0 < t < 1$, linear interpolation gives

$$L(t) = (1 - t)f_0 + tf_1 = f_0 + t\Delta, \qquad \Delta = f_1 - f_0. \tag{1}$$

For an $h = \frac{1}{2}\, 10^{-n}$ we denote generally by $\bar{a}$ the result of rounding off of a to n decimals, so that we have generally

$$|a - \bar{a}| \leqq h, \qquad \bar{a} \equiv 0 \pmod{2h}. \tag{2}$$

We use the notations

$$f_i = \bar{f}_i + \epsilon_i, \qquad |\epsilon_i| \leqq h, \qquad (i = 0, 1), \tag{3}$$

$$\bar{\Delta} = \overline{f_1 - f_0}. \tag{4}$$

With $\bar{\Delta}$ we form

$$L^* = \bar{f}_0 + t\bar{\Delta}, \tag{5}$$

$$L - L^* = \epsilon_0 + t(\Delta - \bar{\Delta}). \tag{6}$$

We consider now for a constant η with $0 \leqq \eta < 1$ in the ϵ_0, ϵ_1-plane the set $S(t, \eta)$ of points for which we have

$$|L - L^*| > (1 + \eta)h,$$

that is,

$$|\epsilon_0 + t(\Delta - \bar{\Delta})| > (1 + \eta)h, \tag{7}$$

where

$$\begin{cases} \Delta - \bar{\Delta} \equiv \epsilon_1 - \epsilon_0 \pmod{2h}, \\ \qquad |\Delta - \bar{\Delta}| < h. \end{cases} \tag{8}$$

Our problem is to determine the area of $S(t, \eta)$. The inequality (7) can be satisfied only for $t > \eta$, since $|\epsilon_0| \leqq h$, $|\Delta - \bar{\Delta}| \leqq h$.

We have

$$|\epsilon_0| \leqq h, \qquad |\epsilon_1| \leqq h. \tag{9}$$

Further we must have

$$|\epsilon_1 - \epsilon_0| > h,$$

since otherwise $\bar{\Delta}$ would be $\bar{f}_1 - \bar{f}_0$ and in that case the left side of (7) is $\leqq h$.

We will have therefore two cases accordingly as

$$\text{I. } \epsilon_1 - \epsilon_0 > h \qquad \text{or} \qquad \text{II. } \epsilon_1 - \epsilon_0 < -h.$$

In both cases we have

$$\epsilon_1\epsilon_0 < 0. \tag{10}$$

Case I. $h \geqq \epsilon_1 > 0 > \epsilon_0 \geqq -h,\ \epsilon_1 - \epsilon_0 > h.$
Here we have from (8)

$$\Delta - \bar{\Delta} = \epsilon_1 - \epsilon_0 - 2h,$$

and (7) becomes

$$\epsilon_0 + t(\epsilon_1 - \epsilon_0) - 2th < -(1 + \eta)h, \tag{11}$$

since the left hand expression in (11) cannot be $>h$.

The points satisfying (11) lie beneath the straight line

$$(1 - t)\epsilon_0 + t\epsilon_1 + (1 + \eta - 2t)h = 0. \tag{12}$$

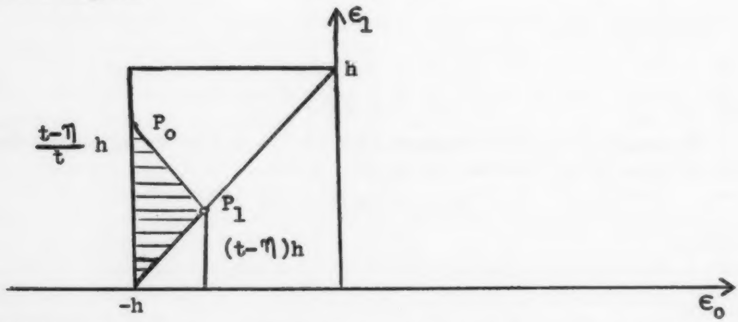
The line (12) meets $\epsilon_0 = -h$ in the point

$$P_0(\epsilon_0 = -h,\ \epsilon_1 = (1 - \eta/t)h)$$

and the line $\epsilon_1 - \epsilon_0 = h$ in the point

$$P_1(\epsilon_0 = (t - \eta - 1)h,\ \epsilon_1 = (t - \eta)h).$$

The corresponding part of the set $S(t, \eta)$ is therefore the shaded triangle in the diagram



and has the area $\frac{1}{2}(t - \eta)^2h^2/t$. In the case II we have an entirely symmetric argument and have therefore for the total area of $S(t, \eta)$ the expression $(t - \eta)^2h^2/t$, while all points (ϵ_0, ϵ_1) satisfying (9) cover the area $4h^2$.

We see that the "geometric" probability that the point (ϵ_0, ϵ_1) will fall into the set $S(t, \eta)$ is $\frac{1}{4}(t - \eta)^2/t$, an expression that is not only positive for $t > \eta$, but even fairly large.

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This work was performed on a National Bureau of Standards contract with the American University.

[1] A. Ostrowski, *Vorlesungen über Differential und Integralrechnung*. Basel, 1951, v. 2, p. 294–296.

Mechanical Quadrature Near a Singularity

The purpose of this note is to present coefficients to facilitate computation of integrals of the type $\int_0^b x^{-\frac{1}{2}}f(x)\,dx$. If the integrand $x^{-\frac{1}{2}}f(x)$ has no singularity at the origin, the coefficients presented are still applicable, though in this case other well-known procedures for numerical integration might be more appropriate. In this regard, it should be noted that suitable modifications of $f(x)$ permit one to employ the coefficients in evaluating $\int_0^b x^{\frac{1}{2}}f(x)\,dx$. In applied problems, the analytical form of $f(x)$ is usually known though it may be rather complicated in form. The singularity may be removed by imposing the transformation $x = y^2$ or by fractional integration after which any one of a number of numerical procedures can be applied. The former approach is not satisfactory since the upper limit of integration as a rule becomes irrational and so complicates tabulation of the resulting integrand. Integration by parts requires the computation of $f'(x)$. If $f(x)$ is quite complicated, its derivative is usually even more so, and again an excessive amount of tabulation is required. A third possibility is to expand $f(x)$ into a power series about the origin. However, in many cases this is not easy owing to the complexity of $f(x)$. Furthermore, the series may be slowly convergent. A more satisfactory procedure is to write the Lagrangian polynomial which precisely fits $f(x)$ at equally spaced points over the interval $(0, b)$. Then a numerical integration formula is easily evolved by calculating the moments $\int_0^b x^{-\frac{1}{2}}x^m dx$, $m = 0, 1, 2, \cdots$. It is convenient to consider the integral in the form $A_n = \int_0^{nh} x^{-\frac{1}{2}}f(x)\,dx$. Let $f_r = f(rh)$, $r = 0, 1, \cdots, n$ and write

$$(1) \qquad A_n = 2(nh)^{\frac{1}{2}}\left(\sum_{r=0}^{n} f_r\gamma_r^{(n)}\right)D_n^{-1} + R_n,$$

where R_n is the remainder term. Following the procedure described above, the exact coefficients $\gamma_r^{(n)}$ and D_n are given in Table I for $n = 1(1)10$.

It is clear that the $(n + 1)$ point formula so derived is exact if $f(x)$ is a polynomial of the nth degree. If $f(x)$ is not of this form, it is of interest to examine the remainder. Evaluation of the remainder term in linear methods of approximation has been discussed by MILNE.[1,2] Following his notation, the remainder is given by

$$(2) \qquad R_n = \int_0^{nh} f^{(n+1)}(s)G(s)\,ds.$$

The analytical form of $G(s)$ is easily deduced from the approximating formula. For the $(n + 1)$ point formula

$$(3) \qquad n!G(s) = \int_s^{nh} x^{-\frac{1}{2}}(x - s)^n dx - 2(nh)^{\frac{1}{2}}\sum_{r=0}^{n}\overline{(rh - s)}^n\gamma_r^{(n)}/D_n,$$

where

$$(4)\qquad \overline{(rh - s)}^n = (rh - s)^n \text{ if } rh \geq s$$
$$= 0 \qquad \text{if } rh < s.$$

The formula for A_n is exact if $f(x) = (x - s)^n$ and so

$$(5)\qquad 0 = \int_0^{nh} x^{-\frac{1}{2}}(x - s)^n dx - 2(nh)^{\frac{1}{2}} \sum_{r=0}^{n} \overline{(rh - s)}^n \gamma_r^{(n)}/D_n.$$

Suppose $0 \leq s \leq h$. Subtracting (5) from (3) and using (4), a straightforward reduction gives

$$(6)\qquad n!G(s) = 2(-1)^n(nh)^{\frac{1}{2}}\gamma_0^{(n)}s^n - \frac{2^{n+1}(-1)^n n! s^{n+\frac{1}{2}}}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n+1)} \text{ if } 0 \leq s \leq h.$$

Denote the right hand side of (5) by $G_1(s)$. Let

$$(7)\qquad \begin{aligned} H_r(s) &= 2(nh)^{\frac{1}{2}}(rh - s)^n\gamma_r^{(n)}/D_n; \\ G_r(s) &= G_{r-1}(s) + H_{r-1}(s), \qquad r = 2, 3, \cdots, n. \end{aligned}$$

Then

$$(8)\qquad n!G(s) = G_r(s) \text{ if } (r-1)h \leq s \leq rh, \quad r = 1, 2, \cdots, n.$$

We extend some terminology by Milne and say that $G(s)$ is definite if it does not change sign in the interval of $(0, nh)$; otherwise, it is indefinite. If $G(s)$ is definite, application of the mean value theorem to (2) gives

$$(9)\qquad R_n = f^{(n+1)}(\theta) \int_0^{nh} G(s)\,ds,$$

where $0 < \theta < nh$. In any event, bounds for the error are given by

$$(10)\qquad \begin{aligned} &|R_n| < M \int_0^{nh} |G(s)|ds \quad \text{or} \quad |R_n| < MKnh, \\ &M = \max_{0\leq s\leq nh} |f^{(n+1)}(s)|, \quad K = \max_{0\leq s\leq nh} |G(s)|. \end{aligned}$$

If $G(s)$ is indefinite, it must vanish at least once in the open interval $(0, nh)$ (observe that $G(0) = G(nh) = 0$). Suppose ξ is the one and only one such point. Then the remainder may be composed of two parts.

$$(11)\qquad R_n = f^{(n+1)}(\theta_1) \int_0^{\xi} G(s)ds + f^{(n+1)}(\theta_2) \int_{\xi}^{nh} G(s)ds$$

where $0 < \theta_1 < \xi < \theta_2 < nh$. The extension of this argument for $G(s)$ vanishing at more than one point in the open interval of integration is obvious.

It is known that $\int_0^{nh} G(s)ds$ may be evaluated directly from the approximating formula without a knowledge of $G(s)$. In the present instance

$$(12)\qquad \int_0^{nh} G(s)ds = 2(nh)^{\frac{1}{2}}h^{n+1}\{n^{n+1}/(2n+3) - \sum_{r=1}^{n} r^{n+1}\gamma_r^{(n)}/D_n\}/(n+1)!$$

Employing the remainder term for the polynomial approximation to $f(x)$,

an expression equivalent to (12) is

$$\int_0^{nh} G(s)ds = \int_0^{nh} x^{\frac{1}{2}}(x - h)(x - 2h) \cdots (x - nh)dx/(n + 1)! \tag{13}$$

Thus, if $G(s)$ is definite, computation of a rigorous error term is considerably simplified. As to the definiteness of $G(s)$, no general theorem is available. But employing (8), we can compute $G(s)$ for each formula and ascertain in a heuristic fashion if it is definite. In this study, $G(s)$ is definite for all the even point formulas. For the odd point formulas, $G(s)$ vanishes at one point in the open interval $(0, nh)$. For each integration formula given in Table I, the exact value of $\int_0^{nh} G(s)ds$ is presented in Table II. For the odd point formulas, the values of ξ, $\int_0^{\xi} G(s)ds$ and $\int_{\xi}^{nh} G(s)ds$ are also tabulated; the first mostly to 5D, the latter to 6D.

The coefficients $\gamma_r^{(n)}$ were checked by verifying that (1) is exact for $f(x) = x^m$, $m = 0, 1, \cdots, n$. The values of $\int_0^{nh} G(s)ds$ were obtained employing (12) and checked by (13) and integration of (8). The function $G(s)$

TABLE I

VALUES OF $\gamma_r^{(n)}$ AND D_n

In each column headed by n, the first coefficient is $\gamma_0^{(n)}$, the second is $\gamma_1^{(n)}$, etc. The last number is the value of $D_n = \sum_{r=0}^{n} \gamma_r^{(n)}$.

$n = 1$ (2-point)	$n = 2$ (3-point)	$n = 3$ (4-point)	$n = 4$ (5-point)	$n = 5$ (6-point)	$n = 6$ (7-point)
2	6	34	250	972	15498
1	8	45	416	1685	31032
—	1	18	24	40	−7965
3	—	8	224	840	26480
	15	—	31	460	−3870
		105	—	161	12312
			945	—	1588
				4158	—
					75075

$n = 7$ (8-point)	$n = 8$ (9-point)	$n = 9$ (10-point)	$n = 10$ (11-point)
7 66808	59 61306	548 91535	24888 70076
16 03182	138 56896	1328 43888	65511 43600
−5 35080	−82 58912	−923 11164	−62119 84725
15 04055	203 11680	2220 74370	1 55860 46400
−2 88120	−134 03240	−1681 73334	−1 74234 77400
5 53602	142 49344	1659 86415	1 97949 48768
3 48488	−32 57376	−407 45628	−1 26966 12600
1 01115	44 69632	342 44694	84722 73600
—	5 30095	245 60415	−20655 00900
40 54050	—	61 17959	18057 96400
	344 59425	—	1976 69471
		3394 89150	—
			1 64991 72690

TABLE II

ERROR COEFFICIENTS

n	$\int^{nh} G(s)ds/h^{n+1}(nh)^{1/2}$	ξ/h	$\int_0^{\xi} G(s)ds/h^{n+3/2}$	$\int_{\xi}^{nh} G(s)ds/h^{n+3/2}$
1	−2/15			
2	8/315	1.24092	0.039468	−0.003551
3	−1/35			
4	16/1485	2.69858	0.022193	−0.000644
5	−1018/81081			
6	152/25025	4.11461	0.015041	−0.000163
7	−35098/49 22775			
8	180 95776/45831 03525	5.53190	0.011218	−0.000050
9	−26 20473/5658 15250			
10	1061 15816/3 79480 97187	6.9556	0.008860	−0.000017

was expanded in a Taylor series about an approximate value of ξ, and ξ was determined by inversion of this series. Taylor's formula was also used to calculate $\int_0^{\xi} G(s)ds$. The latter calculations were performed in duplicate and are correct to within one unit of the last decimal place given.

It appears that coefficients of the kind presented here were first derived in a thesis by M. BATES.[3] This work gives exact coefficients for $\int_0^{nh} x^{m/2}f(x)dx$, $m = \pm 1$, $n = 2(1)4$. The remainder terms are examined in some detail. In a thesis by M. E. YOUNGBERG,[4] similar formulas were given for $n = 3(1)7$. With the exception of the seven- and eight-point formulas (the eight-point formula is totally in error), exact coefficients are also tabulated to enable integration over any sub-interval defined by the points at which $f(x)$ is tabulated. No estimates of the error are given.

In a recent paper, E. L. KAPLAN[5] has derived three- and five-point coefficients for the evaluation of $\int_{r_1h}^{r_2h} x^{m/2}f(x)dx$, $m = \pm 1$. All coefficients are in decimal form. For three points, coefficients are given for $r_1 = 0(1)18$. In addition, some coefficients are presented so that, using three ordinates, the integral over the first two or last two ordinates can be found. The five-point case is similarly treated, but the values permit integration over three ordinates at most. Two sets of coefficients are tabulated. One is exact for consecutive powers of x as in our case. The other is exact for $f(x)$ an even function. Finally, error terms are given for selected cases having the former property. In effect, it is assumed that the mean value theorem is applicable and the error coefficients are derived by evaluating equations of the type as of the right hand side of (12). But the author rightly remarks that the results must be used with caution if $r_2 - r_1 > 1$. As a further aid in evaluating the remainder, Kaplan also gives values which account for the second approximated term in the Taylor series expansion of $f(x)$. We have examined the $G(s)$ function for $n = 2$, $r_1 = 4$, $r_2 = 6$ and $m = -1$ and find that it too vanishes in the open interval of integration. The same is true for the three-point formula where $r_1 = 0$, $r_2 = 2$ and $m = 1$.

The author acknowledges with thanks the aid of Dolores Ufford, who assisted in the calculations.

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[1] W. E. Milne, "The remainder in linear methods of approximation," NBS, *Jn. of Research*, v. 43, 1949, p. 501–511.
[2] W. E. Milne, *Numerical Calculus*. p. 108–116.
[3] M. Bates, *On the Development of Some New Formulas for Numerical Integration*. Stanford University, June, 1929.
[4] M. E. Youngberg, *Formulas for Mechanical Quadrature of Irrational Functions*. Oregon State College, June, 1937. (The author is indebted to the referee for references 3 and 4.)
[5] E. L. Kaplan, "Numerical integration near a singularity," *Jn. Math. Phys.*, v. 26, April, 1952, p. 1–28.

On the Numerical Solution of Equations Involving Differential Operators with Constant Coefficients

1. The General Linear Differential Operator. Consider the differential equation of order n

$$Ly + F(y, x) = 0, \tag{1}$$

where the operator L is defined by

$$Ly = \sum_{k=0}^{n} P_k(x) \frac{d^k y}{dx^k},$$

and the functions $P_k(x)$ and $F(y, x)$ are such that a solution y and its first n derivatives exist in $0 \leq x \leq X$. In the special case when (1) is linear the solution can be completely determined by the well known method of variation of parameters when n independent solutions of the associated homogeneous equations are known. Thus for the case when $F(y, x)$ is independent of y, the solution of the non-homogeneous equation can be obtained by mere quadratures, rather than by laborious stepwise integrations. It does not seem to have been observed, however, that even when $F(y, x)$ involves the dependent variable y, the numerical integrations can be so arranged that the contributions to the integral from the upper limit at each step of the integration, at the time when y is still unknown at the upper limit, drop out. Thus again the computation can be made to involve merely quadratures.

It is not often that the solution of the homogeneous equation can be simply determined, and it is perhaps for this reason that attention has not been given heretofore to the possibility of simplifying the numerical evaluation of the solution by making use of the solutions to the homogeneous equation. However, in the case when the functions $P_k(x)$ in L are constants, the solution of the homogeneous equation is easy to determine. This is particularly true when the order of the differential equation is fairly low. In the instance when the operator L is of second order, with constant coefficients, the method of using the integral equation often has decided advantages over the usual methods employed for solving differential equations.

For this reason attention will now be centered on a second order operator with constant coefficients.

2. Linear Differential Operators of Second Order with Constant Coefficients. Let Ly now be specialized as follows:

$$Ly = \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy, \tag{2}$$

where b, c are real constants. Let $y(0) = \rho_0$, and $y'(0) = \rho_1$ be assigned. The differential equation (1) can be replaced by the integral equation

$$y = \alpha_1 e^{m_1 x} + \alpha_2 e^{m_2 x} - G_1(x) + G_2(x), \tag{3}$$

where

$$\alpha_1 + \alpha_2 = \rho_0; \quad \alpha_1 m_1 + \alpha_2 m_2 = \rho_1,$$

$$G_k(x) = \left[e^{m_k x}\int_0^x e^{-m_k t}F(y, t)dt\right]\Big/(b^2 - 4c)^{\frac{1}{2}}; \quad k = 1, 2,$$

and m_1, m_2 are the roots of $m^2 + bm + c = 0$, provided $b^2 - 4c \neq 0$. In the special case when $b^2 - 4c = 0$, (3) becomes

$$y = e^{-\frac{1}{2}bx}\left[\alpha_1 + \alpha_2 x + \int_0^x (t - x)e^{\frac{1}{2}bt}F(y, t)dt\right], \tag{3a}$$

$$\alpha_1 = \rho_0, \quad \alpha_2 = \rho_1 + \tfrac{1}{2}b\rho_0.$$

It should be observed that when $b^2 - 4c$ is negative, m_1 and m_2 are conjugate complex numbers. When b, c, and the initial values are real, the imaginary component of (3) will drop out, and the discussion which is to follow will apply to this case and to (3a) as well. For the sake of simplicity, therefore, we shall now assume that m_1 and m_2 are real and distinct. Let

$$x_r = x_0 + rh, \quad x = x_0 + sh, \quad t = x_0 + ph, \quad y_r = y(x_r).$$

With the above, $G_k(x)$ takes the form

$$G_k(x) = e^{m_k(x - x_0)}G_k(x_0) + C_k(x), \tag{4}$$

where

$$C_k(x) = e^{m_k sh}h(b^2 - 4c)^{-\frac{1}{2}}\int_0^s e^{-m_k ph}F(y, x_0 + ph)dp. \tag{5}$$

If the integrand of (5) is approximated by an $(s + 1)$-point polynomial, then

$$\int_0^s e^{-m_k ph}F dp = \sum_{r=0}^{s} a_r e^{-m_k rh}F(y_r, x_r) + R_k, \tag{6}$$

where the coefficients a_r result from the integration of the polynomial and the truncation term, R_k, can be represented by

$$R_k = \int_0^s \phi(t)[t, x_0, x_1, \cdots, x_s]dt. \tag{7}$$

In (7) $[t, x_0, x_1, \cdots, x_s]$ is the divided difference[1] of order $(s + 1)$ of the function $e^{-mt}F(y, t)$ and $\phi(t)$ is the polynomial approximation of $e^{-mt}F(y, t)$.

Thus

$$(8)\quad -G_1(x) + G_2(x) = -e^{m_1(x-x_0)}G_1(x_0) + e^{m_2(x-x_0)}G_2(x_0) + h(b^2 - 4c)^{-\frac{1}{2}} \sum_{r=0}^{s-1} aF(y_r, x_r)\{e^{m_2h(s-r)} - e^{m_1h(s-r)}\} - R_1 + R_2.$$

Note that the term involving $F(y_s, x_s)$ dropped out, since the coefficient of this term appeared with the same sign in G_1 and G_2. Although this fact is well known in the theory of integral equations, its importance from the viewpoint of the numerical evaluation of the solution needs emphasis. Thus the evaluation of $-G_1 + G_2$ *does not depend on the value of the function at the end of the interval.* We therefore do not need a "predictor" formula (using MILNE's terminology[2]). The steps of integration at any stage can therefore be carried out as follows:

1) Evaluate $-G_1(x) + G_2(x)$ by (8).
2) Compute $\alpha_1 \exp(m_1x) + \alpha_2 \exp(m_2x)$; hence knowing $(-G_1 + G_2)$, we now know y_s, from (3).
3) Knowing y_s, we can use (6) to evaluate the integral $G_1(x)$ by a mere quadrature. $G_2(x)$ is now also known, since $G_1(x)$ and $-G_1(x) + G_2(x)$ are known.

3. The Truncation Term. If m_1 and m_2 are negative, $e^{-m_1t}F$ may require a higher order approximation formula than $F(t)$ itself (although by no means necessarily so). In any case, h must be small enough so that R_k in (7) is indeed negligible for the accuracy aimed at. In some cases it is actually possible to take a *larger* step h when the integral equation is used, than the one that can be taken when the differential form (2) is operated with. Moreover, the process of solving the integral equation may be more stable than the corresponding solution of the differential equation by stepwise integration. This is especially true when $(b^2 - 4c)^{\frac{1}{2}}$ is large numerically. The example following illustrates the case.

4. Example. Consider the differential equation

$$(9)\qquad u'' + vu' + e^{-x} + f(u) = 0,$$

where

$$f(u) = \exp\left(A - \frac{B}{u + d}\right).$$

This differential equation occurred in connection with certain steady state solutions, and among the various parameters which were used, one set was the following:

$$A = 74.997736;\quad B = 257.42325;\quad d = 2.19885;\quad u(0) = 1.294;$$
$$u'(0) = -M = -10;\quad v = 24.38.$$

Suppose we attempted the evaluation of (9) by the usual method of stepwise integration, first computing an integral of u'', using (9), then integrating u' to obtain u. An analysis of the manner in which an error in u' is propagated shows that the interval h would have to be taken no larger than $1/(5v)$ over the entire range of the integration. Thus for $v = 25$, an interval as small as 0.008 would be needed.

A study of the behavior of $f(u)$, which depends not only on v but also on M and $u(0)$, shows that for very small values of x, it is sometimes necessary to take h even smaller than $1/(5v)$. This is true whether we solve the equation by integrating (9) or by using (3). However, when $M = 10$, for example, and the integral equation is used, it is possible to use a step h as large as 0.1, for x larger than 0.4, and yet we would have to maintain an interval of about 0.008 if the differential equation were used. It can be verified that it is possible to lose all significance in u' within a relatively short range of x when (9) is used, unless the interval h is kept small enough. When (3) is used, on the other hand, the solution remains very stable, and the size of the interval can be increased just as quickly as the behavior of $f(u)$ permits. It turns out that $e^x f(u)$ behaves no worse than $f(u)$ itself. The successive derivatives of $f(u)$ with respect to x are numerically very large near the origin, but they go down in magnitude to reasonable levels at $x = 0.4$, where $M = 10$. For this problem, therefore, the form (3) is far superior to the usual method of solving the differential equation.

The permissible magnitude of h near the origin, when (3) is employed, can be determined by usual methods, and the necessary starting values can be computed in a simple manner; hence it is not deemed worth while to give the specific details of the solution. We shall, however, re-write (3) to apply specifically to (9). Here

$$F = f(u) + e^{-x}; \quad m_1 = 0; \quad m_2 = -v; \quad \rho_0 = u(0); \quad \rho_1 = -M.$$

When the above are substituted into (3), the equation becomes

$$u = U_1 - \frac{1}{v}\int_0^x f\{u(t)\}\,dt + \frac{1}{v}e^{-vx}\int_0^x e^{vt}f\{u(t)\}\,dt,$$

where

$$U_1 = \frac{e^{-x}}{v-1} + \frac{e^{-vx}}{v}\left[M - \frac{1}{v-1}\right] + u_0 - \frac{M+1}{v}, \quad v(v-1) \neq 0.$$

Thus let

$$G(x) = \frac{1}{v}\int_0^x f\,dt; \quad S(x) = \frac{1}{v}e^{-vx}\int_0^x e^{vt}f\,dt - \frac{1}{v}\int_0^x f\,dt,$$

$$H(x) = \frac{1}{v}e^{-vx}\int_0^x e^{vt}f\,dt = S(x) + G(x).$$

Using the notation

$$\phi_s = \phi(sh)$$

for any function ϕ, we have, by the previous analysis,

$$u_{s+1} = U_{s+1} + \frac{1}{v}e^{-2vh}[S_{s-1} + G_{s-1}] - \frac{1}{v}G_{s-1} + \phi_{s+1},$$

where

$$\phi_{s+1} = \frac{1}{v}e^{-v(s+1)}\int_{(s-1)h}^{(s+1)h} e^{vt}f\,dt - \frac{1}{v}\int_{(s-1)h}^{(s+1)h} f\,dt.$$

If ϕ_{s+1} is evaluated by Simpson's rule, we have

$$\phi_{s+1} = \frac{h}{3v}[4(e^{-vh} - 1)f_n + (e^{-2vh} - 1)f_{s-1}] + R + \delta_{s+1},$$

where δ_{s+1} is the rounding error. The total error at x can be approximated, roughly, by

$$\epsilon_{s+1} = -\frac{1}{v}\int_0^{x_{s+1}} (\delta + R)\frac{\partial f}{\partial u}\,dt + \frac{1}{v}e^{-vx_{s+1}}\int_0^{x_{s+1}} e^{vt}(\delta + R)\frac{\partial f}{\partial u}\,dt.$$

Differences can be used in the usual manner to approximate the magnitude of derivatives. The recommendations for the special example are:

a) Compute ϕ_{s+1} by Simpson's rule or some other suitable integration formula.
b) Evaluate u_{s+1}.
c) Evaluate G_{s+1} by quadrature (since u_{n+1} is now known to the required accuracy).
d) Write $S_{s+1} = u_{s+1} - U_{s+1}$.

Return to (a) for the evaluation of ϕ_{s+2} in the succeeding step.

The process can be readily coded for high-speed machines.

One may inquire whether (3) is always the better form to use, compared with (1), or whether there are ranges of the parameters where one of the forms is better than the other. An examination of the way in which the constants enter into the solution shows that when $b^2 - 4c$ is large form (3) would always have advantages over (1), since the square root of this quantity enters into the denominator of some of the terms. In the case when this quantity is small, however, it is likely that (1) would be the better form to operate with.

5. The General Case. It may be worth while to remark that the method applies to the general case mentioned in Section I, where the coefficients $P_k(x)$ are functions of x. The solution can be *constructed formally* by using the usual method of variation of parameters for linear equations.[3] The complete solution will be of the form

$$y = \sum_{k=1}^{n} u_k V_k,$$

where the n functions u_k are the known solutions of the homogeneous equation, and the n integrals V_k must be generated numerically by using the Wronskian of the solutions u_k. In the numerical process, the value of y at some point x is obtained from the property that the contribution to $y(x)$ from the upper limit of integration drops out. After $y(x)$ is known, each individual $V_k(x)$, which is required for carrying forward the numerical process, is then evaluated by a mere quadrature.

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[1] See L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, London, 1933.
[2] W. E. MILNE, *Numerical Calculus*, Princeton, 1949.
[3] E. L. INCE, *Ordinary Differential Equations*, London, 1927, section 5.23.

RECENT MATHEMATICAL TABLES

1018[D].—J. PETERS, *Sechsstellige Werte der Kreis- und Evolventen-Funktionen von Hundertstel zu Hundertstel des Grades nebst einigen Hilfstafeln für die Zahnradtechnik.* Zweite verbesserte Auflage, Bonn, Ferd. Dummlers Verlag, 1951, viii, 222 p., 14 × 20.8 cm.

The first edition of this work (viii, 182 p.) appeared in 1937 and the size of its page was 19.1 × 26 cm. For the second edition a photographic copy of these pages was made in reduced size, the only changes being in the addition on page viii of a preface for the second edition, and of nine new words on the title page. It will be observed, however, that an Appendix of 40 new pages has been added. These pages are mainly filled with tables of results useful in studying problems dealing with teeth of gears.

In the main body of the volume, with α° as argument at interval .01, one may read off on opposite pages 6D values of the 6 trigonometric functions; of the arc α; of the polar coordinates θ° and sec α of the evolute of a unit circle; of the radius of curvature, tan α of the evolute; and of arc $\theta = \text{ev}\,\alpha = \tan\alpha - \text{arc}\,\alpha$.

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1019[F, L].—L. A. DRAGONETTE, "Some asymptotic formulae for the mock theta series of Ramanujan," Amer. Math. Soc., *Trans.*, v. 72, 1952, p. 474–500.

The author is concerned chiefly with the function

$$f(q) = \sum_{n=0}^{\infty} q^{n^2}\{(1+q)(1+q^2)\cdots(1+q^n)\}^{-2}$$

introduced by RAMANUJAN. This function generates the function $A(n)$ defined by

$$f(q) = \sum_{n=0}^{\infty} A(n)q^n,$$

which is similar to the partition function $p(n)$ in that $A(n)$ possesses an asymptotic expansion similar to that of HARDY & RAMANUJAN.[1] Table 1 (p. 495) gives the exact values of $A(n)$ for $n = 0(1)100$. The asymptotic formula is

$$A(n) = \sum_{k<n^{\frac{1}{2}}} \lambda_n(k)(k(n-1/24))^{-\frac{1}{2}} \exp\{(\pi/k)(1/6)(n-1/24)^{\frac{1}{2}}\} + O(n^{\frac{1}{2}}\log n).$$

The coefficients $\lambda_n(k)$ are real and given in Table II (p. 497) as trigonometric polynomials in n for $k = 1(1)14$.

D. H. L.

[1] G. H. HARDY & S. RAMANUJAN, "Asymptotic formulae in combinatory analysis," London Math. Soc., *Proc.*, s. 2, v. 17, 1918, p. 75–115.

1020[G, K].—F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions. Parts II and III," *Biometrika*, v. 38, 1951, p. 435–462.

Part I of this paper appeared in 1949 and is reviewed in *MTAC*, v 4, p. 146, where also will be found references to other symmetric function

tables. The tables of Part I gave expressions for the monomial symmetric functions

$$(\alpha_1, \alpha_2, \cdots) = \sum x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

in terms of the power sums

$$s_r = (r) = \sum x_1^r$$

and inverse tables for all weights ≤ 12.

Part II gives the monomial symmetric functions in terms of the elementary symmetric function

$$a_r = (1^r) = \sum x_1 x_2 \cdots x_r$$

with inverse tables.

Part III gives the monomial symmetric functions in terms of the homogeneous product sums h_r defined by the generator

$$1 + h_1 t + h_2 t^2 + \cdots = (1 - a_1 t + a_2 t^2 - \cdots)^{-1},$$

with inverse tables.

Parts II and III extend also to weights ≤ 12 and have the same format as Part I. The tables of Part II have been compared with earlier tables. The tables of Part III are new.

D. H. L.

1021[G].—M. Osima, "On the irreducible representations of the symmetric group," *Canadian Jn. Math.*, v. 4, 1952, p. 381–384.

A table is given of the number of irreducible representations of the symmetric and alternating groups of order n for $n = 2(1)40$. If a representation by a Young diagram has its rows and columns interchanged we obtain an associated representation. The number of self-associated representations is also given.

D. H. L.

1022[I].—Z. Kopal, P. Carrus, & K. E. Kavanagh, "A new formula for repeated mechanical quadratures," *Jn. Math. Phys.*, v. 30, 1951, p. 44–48.

The authors start with the Hermite interpolation formula

$$(1) \qquad f(x) = \sum_{j=1}^{n} h_j(x) f(a_j) + \sum_{j=1}^{n} \bar{h}_j(x) f'(a_j) + [p_n(x)]^2 f^{(2n)}(u)/(2n)!$$

where

$$\begin{aligned} h_j(x) &= \{1 - (x - a_j)[p_n''(a_j)/p_n'(a_j)]\}[l_j(x)]^2, \\ \bar{h}_j(x) &= (x - a_j)[l_j(x)]^2, \\ l_j(x) &= p_n(x)/[(x - a_j)p_n'(a_j)], \\ p_n(x) &= (x - a_1)(x - a_2) \cdots (x - a_n), \end{aligned}$$

and u is an inner point of the range including all a_j and x. If the coefficients a_j are determined so as to satisfy

$$\int_{-1}^{1} \bar{h}_j(x) dx = 0, \quad j = 1, 2, \cdots, n,$$

then the well-known Gauss quadrature formula results. The authors de-

rived new formulas by setting

$$(2)\qquad \int_{-1}^{1} h_1(x)dx = \int_{-1}^{1} h_j(x)dx = 0, \quad j = 2, 3, \cdots, n.$$

Let

$$\phi(x) = f'(x); \quad \int_{-1}^{1} f(x)dx = 2f(a_1) + U,$$

where

$$U = \int_{-1}^{1} dx \int_{a_1}^{x} \phi(y)dy.$$

When equations (2) are satisfied, there results the quadrature formula

$$(3)\qquad U = 2f(a_1) + \sum_{j=2}^{n} H_j\phi(a_j) + R$$

where the remainder, R, involves the factor $\phi^{(2n-1)}(y)$; hence $R = 0$ if $\phi(y)$ is a polynomial of degree no higher than $2n - 2$. The authors tabulate the weight factors $H_j^{(n)}$ to 6, 7, or 8D, and the points a_j to 7D for $n = 2, 3, 4$, and to 6D for $n = 5$ and 7. They discovered the interesting fact that the system (2) yields more than one set of real solutions a_j for the values of n considered in sufficient detail. The following examples for $n = 3$ illustrate the character of the coefficients:

Type A ($a_1 = 0$)		Type B ($a_1 \neq 0$)	
$a_1 = 0$		$a_1 = \pm.5889711$	
$a_2 = .5477225$	$H_2 = .3042903$	$a_2 = \pm.2250452$	$H_2 = \mp.8445196$
$a_3 = -.5477225$	$H_3 = -.3042903$	$a_3 = \mp.5293628$	$H_3 = \mp.3334225$

No attempt was made by the authors to prove the existence of solutions of (2) for general n; but for odd values of n, solutions of Type A can be shown to exist.

The new formulas have advantages and disadvantages similar to those of the Gauss quadrature formula. It is of course true, as the authors infer, that by a simple linear transformation U can be made to represent the more general double integral

$$W = \int_{c}^{d} dx \int_{b}^{x} \phi(y)dy.$$

Indeed

$$W = U_1 + V_1; \quad U_1 = \tfrac{1}{4}(d - c)^2 \int_{-1}^{1} dw \int_{a_1}^{w} \phi_1(u)du;$$

$$V_1 = \tfrac{1}{4}(d - c)^2 \int_{s}^{a_1} \phi_1(u)du,$$

where

$$\phi_1(u) = \phi[\tfrac{1}{2}(d + c) + \tfrac{1}{2}(d - c)u]; \quad s = (2b - c - d)/(d - c).$$

However, it must be remembered that both V_1 and U_1 need to be evaluated, and that the argument for which ϕ must be computed may be cumbersome in practice. Where the calculations are not prohibitive, a further simplification is in fact possible. For U vanishes whenever $\phi(u)$ is an even function

of u. Hence if $\phi(-y)$ is defined, write in (2)

$$\phi(y) = \tfrac{1}{2}(\phi_3 + \phi_4); \quad \phi_3 = \phi(y) + \phi(-y); \quad \phi_4 = \phi(y) - \phi(-y).$$

It is necessary to apply the quadrature formula only to ϕ_4, if a formula of Type A is used, and thus only half the number of multiplications have to be performed. In many cases arising in practice ϕ_4 may be just as simple to evaluate as ϕ itself.

There is a misprint in the authors' formula (21); in the coefficient of the cosine term replace the term $2x^2$ of the numerator by $2x^3$.

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NBSINA

1023[K].—H. A. DAVID, "Further applications of range to the analysis of variance," *Biometrika*, v. 38, 1951, pp. 393–409.

Consider the mean of k uncorrelated ranges for samples of size n from normal populations with common variance σ^2. Let c^2 be the expected value of $[(\text{mean range})/\sigma]^2$. Let ν equal the number of degrees of freedom for the χ^2-distribution which is approximately equivalent to the distribution of (mean range)/$c\sigma$. Table I contains values of c and ν for $n = 2(1)10$ and $k = 1(1)5, 10$. Table II contains values of c and ν for use in a double classification analysis. Here $n = 2(1)9$ and $k = 2(1)10, 20$. In Table III a split-plot design with l main treatments, m blocks, and N subtreatments (N is reviewer's notation) is considered. Table III contains values of c and $\nu' = \nu/l$ for $m = 2(1)10$ and $N = 2(1)9$. Values of c are given to 2D, those of ν and ν' to 1D. Let d_n be the expected value of (range)/σ for a sample of size n while V_n is the variance of this quantity. Table IV contains values of d_n, d_n/V_n, d_n^2/V_n to 2D for use in analyses of single classification with unequal cell frequencies. Here $n = 2(1)20$.

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1024[K].—P. M. GRUNDY, "The expected frequencies in a sample of an animal population in which the abundances of species are log-normally distributed. Part I," *Biometrika*, v. 38, 1951, p. 427–434.

Given N species in a population with the expected number of individuals per species (abundance), m, following a log normal distribution

$$f(m)dm = \frac{1}{\sqrt{2\pi}m\sigma} e^{-[\ln (m/a)]^2/2\sigma^2}dm,$$

where σ^2 is the variance and $\ln a$ the mean of $\ln m$ (hence a is the median abundance). The probability of obtaining r members of a given species in a sample is $e^{-m}m^r/r!$. Hence the expected proportion of species having r in the sample, ϕ_r, is

$$\phi_r = \int_0^\infty \frac{e^{-m}m^r}{r!} f(m)\, dm.$$

Table 1 presents values of $1 - \phi_0$ (expected proportion of species in the sample) and Table 2 presents values of ϕ_1 (expected proportion of species with singletons) to 4D for $\sigma^2 = 2(1)16$ and $\log_{10} a = -2.0(.25)3.0$.

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1025[K].—P. G. GUEST, "The fitting of polynomials by the method of weighted grouping," *Annals Math. Stat.*, v. 22, 1951, p. 537–548.

Given n equally spaced values of x and the corresponding n observed values of y, it is desired to fit a polynomial $u_p(x) = \sum_{j=0}^{p} b_{pj}x^j$ $(p < n)$ by a method of weighted grouping. To obtain an unbiased estimate of $b_{pp} = a_p$, the n points are divided into not more than $2p + 1$ groups of successive points, the number of points in the ith and in the $(n - i + 1)$th groups being equal. The sum of the y's in each of $p + 1$ of these groups is to be assigned a non-zero weight. Because of symmetry, not more than $(p + 2)/2$ different weights are involved if p is even. If p is odd, $p + 1$ different weights are involved but half of these are the negatives of the other half. The groupings are determined for each n and p in such a way that the variance of a_p, assuming equal variances of the y's, is a minimum. To estimate b_{pj}, $j = 0, 1, \cdots, p - 1$, the a_j, $j = 0, 1, \cdots, p - 1$ are calculated and the estimates of b_{pj} are obtained from the relation, estimate of $b_{pj} = a_j + \beta_{j+1,j}a_{j+1} + \cdots + \beta_{pj}a_p$. (Alternate β's are zero.)

A table (p. 541–545) gives the groupings, the weights (exact), the β's (β_{20} and β_{31}, exact; β_{42} and β_{53}, 10S; β_{40} and β_{51}, 9S) for $n = 7(1)55$, $j = 0(1)5$.

The relative (compared with least square procedure) efficiencies of the estimates of b_{pj} are discussed and a table of some limiting relative efficiencies is given. The lowest limiting relative efficiency appearing in the table is .889 for estimates of a_1. Computational schedules and an example are given.

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1026[K].—H. O. HARTLEY & E. S. PEARSON, "Moment constants for the distribution of range in normal samples," *Biometrika*, v. 38, 1951, p. 463–464.

The moments indicated by the title are the mean μ_1'; the central moments μ_r, $r = 2(1)6$; $\sigma = \sqrt{\mu_2}$; $\beta_1 = \mu_3^2/\mu_2^3$; $\beta_2 = \mu_4/\mu_2^2$; $\kappa_4 = \mu_4 - 3\mu_2^2$; $\kappa_5 = \mu_5 - 10\mu_3\mu_2$; $\kappa_6 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3$. These moments are tabulated for $n = 2(1)20$ except for μ_5, μ_6, κ_4, κ_5, κ_6, which are given for $n = 2(1)12$. The tabled values are given to the number of decimals followed in parentheses by the number of significant digits as follows: μ_1', 5(6); μ_2, 5(5); μ_3, 4(4); μ_4, 3(4); μ_5, 3(3); μ_6, 1(2–3); σ, 4(4); β_1, 4(4); β_2, 3(4); κ_4, 3(2–3); κ_5, 2(0–2); κ_6, 2(0–3).

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1027[K].—L. Kaarsemaker & A. van Wijngaarden, *Tables for Use in Rank Correlation*, Report R 73, Computation Department of the Mathematical Centre, Amsterdam, 1952, 17 p. mimeographed. 21.0 × 32.4 cm.

For a given number n of subjects ranked according to two criteria, the size of Kendall's coefficient of rank correlation is proportional to the score S.[1] If S is the sample value, on the null hypothesis of no association $E(S) = 0$; to test this null hypothesis $P_n(S \geqq \mathrm{S})$, the probability that in n subjects $S \geqq \mathrm{S}$, on the assumption of no association, is used. It is assumed for this paper that no ties in ranks occur. Kendall[2] tabulated $P_n(S \geqq \mathrm{S})$ for $n = 1(1)10$ for all possible values of S and the authors extend this table giving $P_n(S \geqq \mathrm{S})$ to 3D for $n = 1(1)40$. They divide the table into two parts, Tables I and II, for S even and odd, respectively. Table III gives for $n = 4(1)40$ and for $\alpha = .005, .01, .025, .05$ and $.1$, the smallest value of S for which $P_n(S \geqq \mathrm{S}) \leq \alpha$. Since for $n > 40$ the normal distribution is a good approximation to that of S, values of the standard deviation of S, σ_S are listed in Table IV for $n = 40(1)100$ to 5D. There is a typographical error in the formula for σ_S where it is first written on p. 2; it appears correctly on p. 3 and in the table. A recurrence relation used in the calculation of Tables I and II is developed.

C. C. C.

[1] M. G. Kendall, *Rank Correlation Methods*, London, Griffin, 1948.
[2] Loc. cit., p. 141.

1028[K].—K. C. S. Pillai, "Some notes on ordered samples from a normal population," *Sankhyā*, v. 11, 1951, p. 23–28.

Consider a sample of size n from a normal population with zero mean. Table I contains 1% and 5% points of $T = 2$ (midrange)/(range) to 2 and 3D for $n = 3(1)10$. Table II presents values of coefficients for approximate determination of the distribution of the median (average of two central order statistics) to 6D for $n = 4, 6, 8$. Next consider two independent samples from normal populations. Form the range for each sample and let the sample size be n_1 for the larger range and n_2 for the smaller range. Table III contains 1% and 5% points of $F' =$ (larger range)/(smaller range) to 2D for $n_1, n_2 = 2(1)8$. Let σ_1 be the standard deviation for the population yielding the larger range and σ_2 the standard deviation of the other population. Table IV contains a power function comparison of the F'-test and the Snedecor F-test to 3D for $(n_1, n_2) = (3, 7), (4, 4), (8, 2), (5, 2), (8, 3)$, and $\sigma_1/\sigma_2 = 1(.5)2.5$.

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1029[K].—J. E. Walsh, "Some nonparametric tests of whether the largest observations of a set are too large or too small," *Annals Math. Stat.*, v. 21, 1950, p. 583–592.

This paper is apparently the first published work covering nonparametric tests for the detection of outlying observations. For the nonparametric tests, the usual restriction of normality can, of course, be removed and the sample considered to be drawn randomly from one or more con-

tinuous symmetrical populations. Assuming that the sample values are arranged in increasing order of magnitude: $x(1) \leq x(2) \leq \cdots \leq x(n)$, the tests considered are (a) the detection of whether the r largest observations are too large to be consistent with the hypothesis that the populations from which the sample values came have a common median, (b) whether the r largest observations are too small and (c) whether the populations are symmetrical in the tails. With regard to (a) and (b), the proposed tests also cover similar hypotheses for the r smallest observations because of symmetry conditions. The tests, based on the order statistics, covered in the paper have the somewhat fascinating property in that the Type I error or significance level, α, for the sample statistics used is independent of the sample size n for values of n permitted. (Generally speaking, n should be large, in which case the significance level tends toward the value α; however, for no admissible value of n does the significance level exceed 2α.)

The details of the actual tests are rather complicated for description here, but Table 1 gives the necessary specifications for 21 tests and the corresponding significance levels, α, to 4D, for use with $r \geq 4$.

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1030[L].—S. CHANDRASEKHAR, DONNA ELBERT, & ANN FRANKLIN, "The X- and Y-functions for isotropic scattering. I," *Astrophys. Jn.*, v. 115, 1952, p. 244–268.

S. CHANDRASEKHAR & DONNA ELBERT, "The X- and Y-functions for isotropic scattering. II," *Astrophys. Jn.*, v. 115, 1952, p. 269–278.

The functions in question are solutions of the integral equation

$$X(\mu) = 1 + \mu\omega_0 \int_0^1 (\mu + \mu')^{-1}[X(\mu)X(\mu') - Y(\mu)Y(\mu')]d\mu',$$

$$Y(\mu) = e^{-\tau/\mu} + \mu\omega_0 \int_0^1 (\mu - \mu')^{-1}[Y(\mu)X(\mu') - X(\mu)Y(\mu')]d\mu'.$$

Approximations were described in CHANDRASEKHAR's book *Radiative Transfer*, Chapter VIII.

In the first of these papers the approximate solutions were improved by iteration and are tabulated for $0 \leq \mu \leq 1$ for the following values of the parameters: $\omega_0 = .95, .9, .8, .5$, $\tau = .05(.05).25, .5, 1$. The tables are to 4 or 5D, and the difference between the tabulated solution and the corrected second approximation (of Chandrasekhar's book) is also given.

In the second paper, "standard solutions" are defined in the case $\omega_0 = 1$. They satisfy the relations

$$\int_0^1 X_s(\mu)d\mu = 2, \quad \int_0^1 Y_s(\mu)d\mu = 0.$$

The moments of X, Y, X_s, Y_s are called α_n, β_n, α_n^s, β_n^s. The laws of diffuse reflection are expressed in terms of the solutions

$$X^*(\mu) = X_s(\mu) + Q\mu[X_s(\mu) + Y_s(\mu)],$$
$$Y^*(\mu) = Y_s(\mu) - Q\mu[X_s(\mu) + Y_s(\mu)],$$

where

$$Q = -\frac{\alpha_1^e - \beta_1^e}{(\alpha_1 + \beta_1)\tau + 2(\alpha_2 + \beta_2)}.$$

Table 1a gives α_0, β_0, α_0^*, α_1, β_1, and

$$\mathfrak{s} = 1 - \{(2 - \omega_0\alpha_0)\alpha_1 + \omega_0\beta_0\beta_1\}$$

for $\tau = .05(.05).25, .5, 1$ and $\omega_0 = .95, .90, .80, .5$. Table 1b gives $\alpha_0 + \beta_0$, $\alpha_1 + \beta_1$, $\alpha_2 + \beta_2$, α_1^e, β_1^e, $(\alpha_1^e)^2 - (\beta_1^e)^2$, Q, and $\mathfrak{s}$ for the same values of τ and $\omega_0 = 1$.

Table 2 gives X^*, Y^*, X_s, Y_s for $\omega_0 = 1$, $\tau = .05(.05).25, .5, 1$, $\mu = 0(.01)1$.

Table 3 contains certain combinations of the moments.

A. E.

1031[L].—W. Chester, "The reflection of a transient pulse by a parabolic cylinder and a paraboloid of revolution," *Quart. Jn. Mech. Appl. Math.*, v. 5, 1952, p. 196–205.

Table I (p. 204) gives 4D values of

$$4\pi^{-\frac{1}{2}}\int_0^\infty e^{-(\tau+1)r^2}\left\{\frac{4}{\pi}\left[1 - re^{-r^2}\int_0^r e^{z^2}dz\right]^2 + r^2e^{-2r^2}\right\}^{-1}dr$$

and of

$$\tfrac{1}{2}\int_0^\infty e^{-(\tau+1)r}\left\{\left[1 + \tfrac{1}{2}re^{-r}\int_{-r}^\infty z^{-1}e^{-z}dz\right]^2 + \tfrac{1}{4}\pi^2r^2e^{-2r}\right\}^{-1}dr$$

for $\tau = 0, .1, .2(.2)2$. In the last expression the integral with respect to z is meant as a Cauchy principal value.

A. E.

1032[L].—G. Diemer & H. Dijkgraaf, "Langmuir's ξ, η tables for the exponential region of the $I_a - V_a$ characteristic," *Philips Res. Rep.*, v. 7, 1952, p. 45–53.

The integral $\int^\eta [f(x) + m^2]^{-\frac{1}{2}}dx$, where

$$f(x) = e^x(1 + \operatorname{erf} x^{\frac{1}{2}}) - 1 - 2(x/\pi)^{\frac{1}{2}},$$

was tabulated by Freeman[1] in the range $0 \leq \eta \leq 20$, $0 \leq m^2 \leq 20$. Modern microwave diodes and triodes make it desirable to extend these tables considerably. The present paper contains 3S values for $10^{-2} \leq \eta \leq 60$, $0 \leq m^2 \leq 10^5$. The intervals vary. The two tables overlap and there are discrepancies in the overlapping regions.

A. E.

[1] J. J. Freeman, "Noise spectrum of a diode with a retarding field," NBS, *Jn. of Research*, v. 42, 1949, p. 75–88.

1033[L].—S. Goldstein, "On diffusion by discontinuous movements, and on the telegraph equation," *Quart. Jn. Mech. Appl. Math.*, v. 4, 1951, p. 129–156.

The $\gamma(n, \nu)$ may be defined by the recurrence relation $(r + 1)\gamma(n + 1, \nu) = r[\gamma(n, \nu - 1) + \gamma(n, \nu + 1)] - (r - 1)\gamma(n - 1, \nu)$, together with the

initial conditions

$$\gamma(1, 1) = \gamma(1, -1) = \tfrac{1}{2}, \quad \gamma(1, \nu) = 0 \text{ if } \nu \neq \pm 1.$$

The author obtains the generating function of the $\gamma(n, \nu)$, also integral representations and an asymptotic expansion for n large and $\nu^2 = O(n)$. To test the asymptotic expansion, he tabulates (p. 137) 7D values of $\gamma(15, \nu)$ for $r = \frac{1}{2}, 1, 2, 5$ and $\nu = 1(2)15$.

A. E.

1034[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 23: *Tables of the error function and of its first twenty derivatives*. Cambridge, Mass., Harvard University Press, 1952, xxviii, 276 p., 19.5 × 26.7 cm. $8.00.

The functions tabulated in this volume are

$$\phi^{(-1)}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt, \quad \phi^{(n)}(x) = \frac{d^n}{dx^n}\left[\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right], \quad n = 0, 1, 2, \cdots.$$

The $\phi^{(n)}$ are connected with Hermite polynomials.

Tables of these functions are listed in section 15 of the FMR *Index*. Additional tables are mentioned in *MTAC*, v. 3, p. 521, and tables on punched cards are listed in *MTAC*, v. 5, p. 203, 204.

Modern developments in statistics, probability theory, mathematical physics, and noise and communications theory made new tables desirable over a more extensive range with a finer interval, and, most of all, including derivatives of higher orders. The authors of the present work expect that their tables will satisfy all present needs.

Tables I and II give 6D values for $\phi^{(-1)}$ to $\phi^{(10)}$, where x runs from 0 at intervals of .004 up to the point where ϕ ceases to change. The limiting value is .5 for $\phi^{(-1)}$, and zero for all other $\phi^{(n)}$. The last x increases with n, being 4.892 for $n = -1$, and 8.236 for $n = 10$.

Tables III and IV give values to 7S or 6D for $\phi^{(11)}$ to $\phi^{(20)}$, and x runs from 0 at intervals of .002 up to the point where ϕ vanishes to 6D identically. This point is 8.518 for $n = 11$, and 10.902 for $n = 20$.

The introduction contains a chapter (by WARREN L. SEMON) giving a collection of formulas and integral representations for $\phi^{(n)}$ and their connection with Hermite polynomials; a brief section (by the same author) explaining the computation of the tables and the steps taken to eliminate errors; a chapter (by DAVID MIDDLETON) on applications of the functions tabulated in this volume; and a 10D table of all the zeros of all the functions tabulated in this volume. Each section is accompanied by a list of references.

The tables maintain the high standards set by previous volumes of this series.

A. E.

1035[L].—TH. LAIBLE, "Höhenkarte des Fehlerintegrals," *Zeit. angew. Math. Physik*, v. 2, 1951, p. 484–486.

Relief diagrams of erf $(x + iy)$ for (i) $0 \leq x \leq 5$, $0 \leq y \leq 6$, and (ii) $0 \leq x \leq 1.7$, $0 \leq y \leq 2.2$. 4S values of the real and imaginary parts of the first 5 zeros (in the first quadrant) of erf (z).

A. E.

1036[L].—C.-B. LING, "Tables of values of the integrals $\int_0^\infty x^m dx/\sinh^p x$ and $\int_0^\infty x^m dx/\cosh^p x$," *Jn. Math. Phys.*, v. 31, 1952, p. 58–62.

This paper gives values of the above integrals multiplied by the auxiliary factor $p^{m+1}/2^p(m)!$ mostly to 5D for all admissible pairs of p and m, $p = 1(1)8$, $m = 0(1)15$. The reciprocal of the above factor is given exactly or to 6S for the same range. C. W. NELSON[1] has tabulated the first integral to 12D for $m = p = 1(1)40$. Entries common to both tables are in perfect agreement.

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[1] C. W. NELSON, "A Fourier integral solution for the plane-stress problem of a circular ring with concentrated radial loads," *Jn. Appl. Mech.*, v. 18, 1951, p. 173–182.

1037[L].—R. C. LOCK, "The velocity distribution in the laminar boundary layer between parallel streams," *Quart. Jn. Mech. Appl. Math.*, v. 4, 1951, p. 42–63.

The "standard solution," $g(\xi)$, of the Blasius equation $ff'' + 2f''' = 0$ is that solution which has the expansion

$$g(\xi) = -1 + e^{\frac{1}{2}\xi} - \tfrac{1}{4}e^{\xi} + \cdots$$

for $\xi \leq 0$.

Table II (p. 58) gives 4D values of g, g', g'' for $\xi = -\infty, -16(2) - 10(1) - 5(.4).2$, and $.5537 = \xi_0$, where $g(\xi_0) = 0$.

A second solution, $g_1(\xi)$, defined for $\xi > \xi_0$, depends on a parameter $\rho\mu$. It is a solution of the Blasius equation, subject to the initial conditions

$$g_1(\xi_0) = 0, \quad g_1'(\xi_0) = g'(\xi_0), \quad g_1''(\xi_0) = (\rho\mu)^{\frac{1}{2}}g''(\xi_0).$$

With $a = [g_1'(\infty)]^{-\frac{1}{2}}$, $b = \xi_0$, two further functions f_1 and f_2 are defined by

$$f_1(\eta_1) = ag_1(a\eta_1 + b), \quad f_2(\eta_2) = ag(a\eta_2 + b).$$

These are also solutions of the Blasius equation, $\eta_1 > 0$, $\eta_2 < 0$.

Tables III–VI (p. 59–62) give 4D values of $f_1, f_1', f_1'', f_2, f_2', f_2''$ for, respectively, $\rho\mu = 5.965 \times 10^4, 100, 10, 1$. η_1 and η_2 range over their respective ranges at intervals suited to the physical problem.

Table VII (p. 63) gives, for the case $\rho\mu = 1$, a similar tabulation of that solution for which $f_2'(-\infty) = .5014$ (instead of 0 as in the previous cases).

A. E.

1038[L].—B. MISHRA, "Wave functions for excited states of mercury and potassium," Cambridge Phil. Soc., *Proc.*, v. 48, 1952, p. 511–515.

The radial wave function $P(nl|r)$ for excited state (n, l) of Hg satisfies the differential equation

$$\left\{\frac{d^2}{dr^2} + \epsilon(nl) + \frac{2Z_p(r)}{r} - \frac{l(l+1)}{r^2}\right\}P(nl|r) = 0,$$

where r is the radial distance, $\epsilon(nl)$ is an energy parameter, and $Z_p(r)$ is the total effective nuclear charge. The wave function is normalized so that

$$\int_0^\infty [P(nl|r)]^2 dr = 1.$$

For the numerical integration the variables $\rho = \ln(1000\,r)$ and $S = r^{-\frac{1}{2}}P$ were used.

Table 1, p. 512, gives 3 to 5S values of Z_p and 4D values of S for the states $(6s)$, $(6p)$, $(6d)$, $(7s)$, $(7p)$, $(7d)$, for $\rho = 0(1/3)2(1/6)4(1/12)11{\cdot}25$. The values of Z_p were derived from existing tables for the neutral atom, and the wave functions found by numerical integration of the differential equation.

The differential equation for K contains an additional term $V_p(r) = -\frac{1}{2}\alpha(r^2 + r_0^2)^{-2}$. Table 3 gives 4D values of $P(4p|r)$ for $r = 0(.02).3(.05).6(.1)1.2(.2)3.2(.4)16(1)30$.

Tables 2 and 4 give the numerical values of certain integrals.

A. E.

1039[L].—A. Papoulis, "On the accumulation of errors in the numerical solution of differential equations," *Jn. Appl. Phys.*, v. 23, 1952, p. 173–176.

By solving a differential equation with various initial conditions the author exhibits a method for determining the error in the numerical solution of such equations. Consider the differential equation (†) $y' = f(x, y)$ and assume that the interval $[a, b]$ is divided into n subintervals by the points of subdivision x_k. Let β_k be the difference between the value of the correct solution at $x = x_k$ and the computed solution y_k at x_k. (β_k is the sum of truncation and round-off errors.) If we assume that $y(x)$ is correct up to x_{k-1} and from x_k to b satisfies (†), then at $x = b$ we shall have an error $E_k = y(b) - y_n$. Let $y(x)$ be a solution of (†) with the initial condition $y(x_0) = y_0$ and $Y(x)$ a solution of (†) with $Y(x_0) = y_0 + M$ (M a constant); then under the assumption (A): $r(x) = Y(x) - y(x)$ is small compared with $y(x)$, the author shows that the total error R is the sum of the E_k $(= \beta_k[r(b)/r(x_k)])$, and hence for n large,

$$R = [nr(b)/(b-a)] \int_a^b [\beta(x)/r(x)]dx.$$

Under assumption (A), $f(x, Y(x)) - f(x, y(x)) \doteq r(x)\varphi(x)$, where $\varphi(x) = \partial f(x, y(x))/\partial x$ and hence $r(x)$ is determined from the differential equation $r'(x) = \varphi(x)r(x)$. If $\beta(x)$ is the result of round-off errors alone and if they are random with variance d, then the dispersion of R is

$$[d^2nr^2(b)/(b-a)] \int_a^b [1/r^2(x)]dx.$$

Extensions to systems of two equations, $dx/dt = f(x, y, t)$, $dy/dt = g(x, y, t)$, are given in detail with some remarks on systems of higher order.

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1040[L].—A. VAN WIJNGAARDEN, *Table of the integral*

$$\int_0^1 \exp(-v^{-2} - xv)v^{-p}dv.$$

Mathematical Centre, Amsterdam, *Report* R 176, 1952, 6 mimeographed leaves.

The integral mentioned in the title is given to 5D ("the last decimal is not absolutely reliable") for

$$x = 0(.25)3.5(.5)14.5,$$
$$p = 0(1)10.$$

For $x = 0$ the integral becomes

$$\tfrac{1}{2}\int_1^\infty t^{(p-3)/2}e^{-t}dt,$$

an incomplete gamma function. A special table for $x = 0$ to 6D is given for $p = -25(1)11$. Several methods were used to compute the tables including power series and recurrence relations resulting from integration by parts.

D. H. L.

1041[L].—E. M. WILSON, *Solutions of the Equations* $(y'')^2 = yy'$ *and Two Other Equations*, Admiralty Research Laboratory, Teddington, Middlesex, November, 1951.

The first part of this report is concerned with the solution of $y'' = -\sqrt{yy'}$ such that $y(0) = 0$, $y'(0) = 1$. Solution depends on the inversion of the integral $x = \int_0^y (1 - t^{\frac{3}{2}})^{-\frac{1}{2}}dt$, where x is the independent variable. A generalization of this integral has been studied by R. GRAMMEL (*MTAC*, v. 5, p. 155). When x approaches $4\pi\sqrt{3}/9$, y approaches unity while y' and y'' approach zero. Starting at this point the function tabulated is the solution of $y'' = +\sqrt{yy'}$ which admits of an integral representation similar to the previous one. Series solutions are also given. A photostat table is available giving y to 6D for $x = 0(.002)6$ and $\log y$ to 6D for $x = 6(.01)7$. The report tabulates y to 6D for $x = 0(.05).5(.1)6$. Author believes maximum error is less than 0.7 unit in the last figure given. Second differences, mostly modified, are also tabulated. To facilitate interpolation near the origin, $y = 4x^{\frac{5}{2}}/15$ is tabulated to 6D for $x = 0(.05).25$.

In the second part, values of the integral

$$f(\beta, \rho) = (2e^{-\rho^2}/\beta^2\sqrt{\pi})\int_0^\beta I_0(2\rho\eta)e^{-\eta^2}\eta d\eta$$

are tabulated to 4D for $\beta = 0(.25)4$, $\rho = 0(.25)5$. I_0 is the modified Bessel function. The author claims entries are correct to within one unit of the last figure. The table has been subtabulated so that entries are in intervals of .05 and is available in photostat form. A more extensive tabulation of a function simply related to the above has been made by S. R. BRINKLEY, JR. & R. F. BRINKLEY (*MTAC*, v. 2, p. 221) and S. R. BRINKLEY, JR., H. E. EDWARDS & R. W. SMITH, JR. (*MTAC*, v. 6, p. 40).

Part three tabulates to 3D that zero of $u \sin x - \cos x + e^{-ux}$ which lies between π and 2π for $u = .1(.01).3(.02)2$ and $\sqrt{u} = 0(.02).5$. The author states that the last figure should be correct to within 0.7 of a unit. Linear interpolation yields full accuracy and first differences are provided.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to errata in RMT **1022**, **1032**.

214.—E. P. ADAMS & R. L. HIPPISLEY, *Tables of Elliptic Functions, Smithsonian Miscellaneous Collections*, v. 74, no. 1, Washington 1939, 1947.

The heading of page 294

for $E' = 1.5629622295$
read $E' = 1.5631622295$.

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215.—(1) A. A. GERSHUN, "Berechnung des Volumleuchtens," *Physikalische Z. d. Sowjetunion*, v. 2, 1932, p. 149–185 [*MTAC*, v. 2, p. 191].

(2) E. SCHMIDT, "Die Berechnung der Strahlung von Gasräumen," *Zs. Verein Deutscher Ingenieure*, v. 77, 1933, p. 1162–1164.

(3) S. GOLDSTEIN, "On the vortex theory of screw propellers," Roy. Soc., London, *Proc.*, v. 123A, 1929, p. 440–465.

Three tables are given in (1), on p. 172, 175, and 180, respectively.

Table I, containing the function

$$F_1(x) = 1 - (1 - x)e^{-x} + x^2 \operatorname{Ei}(-x) = 1 - 2x^2 \int_x^\infty \frac{e^{-t}}{t^3}\, dt$$

to 4D for $x = 0(.01).02, .05, .1, .2(.2)1, 1.6(.4)2.4, 3, 4$, was read against the same function given in (2) in complementary form on p. 1163, also to 4D mainly, for $x = 0(.01).02, .05, .1(.1)1.0(.2)2.0, 2.4, 2.5, 3(1)5$. The discrepancies, and the extra values given in (2), were checked, revealing the following errors in (1) and (2):

	F_1	*for*	*read*
(1)	0.4	.4925	.4854
	3.0	.9822	.9821
(2)	0.3	.6000	.6001
	0.9	.2516	.2514
	1.2	.1680	.1679
	1.4	.1296	.1292
	1.6	.1011	.0998
	1.8	.0777	.0774
	2.5	.0328	.0326
	4.0	.00545	.00552
	5.0	.00175	.00176

Table II, of the function

$$F_2(x) = 2\sum_{1}^{\infty} \frac{(-1)^{p+1}x^p}{(p+2)p!} = 1 - \frac{2}{x^2}(1 - (1+x)e^{-x}),$$

was recomputed. The following values are in error:

F_2	*for*	*read*
2.8	.8039	.8038
4.0	.8863	.8864
5.0	.9289	.9232

Table III gives $F_x = 1 - \int_0^{\pi/2} e^{-x\cos t}\cos t\,dt = \frac{1}{2}\pi[I_1(x) - L_1(x)]$,

where I_1 and L_1 are Bessel and Struve functions of imaginary argument. Comparison with (3) indicated one error in (1)

$x = 4$	*for* 9281	*read* 9271

and two in (3)

$\mu = 0.8$	*for* 0656	*read* 0671
1.2	*for* 0039	*read* 0045

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216.—P. R. E. JAHNKE & F. EMDE, *Tables of Higher Functions*. 4th (revised) ed. Leipzig, 1948.

This edition gives on p. 9 a more extensive table of the maxima and minima of the sine integral, that is, si (x) for $x = \pi(\pi)24\pi$. Comparing these values with those obtained from interpolating in the NBS tables we find the seven errata:

$\pi^{-1}x$	*for*	*read*
18	−0.007673	−0.017673
19	+0.006744	+0.016744
20	−0.005907	−0.015907
21	+0.005151	+0.015151
22	−0.004463	−0.014463
23	+0.003834	+0.013834
24	−0.008258	−0.013258

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On p. 239 the entry immediately to the right of 6.3 in the x column should be shifted down one line. Thus $-H_1^{(1)}(i6.30) = 0.0^36170$. The first significant figure of the function does not change from 6 to 5 until x is between 6.32 and 6.33.

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217.—NILS PIPPING, "Die Goldbachsche Vermutung und der Goldbach-Vinogradowsche Satz," Åbo, Finland, Akad., *Acta Math. Phys.*, v. 11, no. 4, 1938, p. 1–25.

x	*for* m_x	*read*
6944	61	37
10006	149	83
23926	47	17
31004	73	67

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UNPUBLISHED MATHEMATICAL TABLES

In this issue there is a reference to an unpublished table in RMT **1041**.

150[F].—D. D. WALL, *Table of Wilson's Quotient*. 11 leaves tabulated from punched cards. Deposited in the UMT FILE.

For each of the 709 primes $p \leq 5381$ the table gives the least positive remainder on division of $\{(p-1)! + 1\}/p$ by p. This remainder is zero for $p = 5, 13$, and 563. The table was produced on the IBM Card Programmed Calculator. [See also *MTAC*, v. 5, p. 81, MTE **182**.]

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AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 415 South Building, National Bureau of Standards, Washington 25, D. C.

DISCUSSIONS

ASYNCHRONOUS SIGNALS IN DIGITAL COMPUTERS

It is frequently necessary, during the operation of a digital computer, to inject signals from sources that are not synchronized with the computer itself, for example, the manual signals. This operation may be initiated by pressing an appropriate push button. In this discussion, we will not be concerned with such problems as "bounce" of contacts, wavering pressure or the possibility of repeated operation because of completion of computation before the button is released, but only with the fact that the contact is made (or broken) at a random moment with respect to the computer timing pulses or "clock." Probably the most important source of automatically generated signals asynchronous with the computer proper is the input equipment. Whether data are introduced by magnetic tape, punched cards, manual keyboard or other means, it is generally introduced at a much lower rate than transfers within the computer itself and at intervals which do not synchronize with the main "clock."

The presence of asynchronous signals creates a problem of special type in that their effect under certain conditions is not strictly determined in the digital sense. That is, examination of such a signal at a time synchronous with the machine proper may not be interpretable as either "yes" or "no" over a narrow range of relative timing of the signal to the machine, but only as "maybe." If an entity similar to one of Maxwell's famous demons were available to convert such "maybe's" to either "yes" or "no," even in arbitrary fashion, the problem would be solved. Automatic equipment, however, responds in continuous fashion to signal amplitudes and durations, and these must have a transition range between any discrete set of states. When one is dealing with wholly synchronous signals, such transitions can be made to take place between examining times and thus do not cause difficulty. With asynchronous signals, the random relative timing causes randomness in the transition durations and thus permits conditions under which examination of the signal may occur during the period of transition.

One might reply to the above discussion that the existence of an indefinite "maybe" is not of much importance since, no matter whether the remaining equipment behaves as if it were "yes" or as if it were "no," operation would be satisfactory. This is true if *all* the remaining equipment made the same interpretation of the "maybe," and, in fact, in certain cases it suffices to "weight" one element deliberately so that its interpretation of the "maybe" can never be "yes" when any of the other elements behave as if the "maybe" is "no." In most cases, however, error results if any two elements do not interpret the signal in the same manner. The "maybe" condition thus usually implies error. An otherwise perfect machine therefore is not error free in the presence of asynchronous signals, but has a finite probability of error. The problem is to make this probability extremely small.

This problem was recognized at an early stage in computer development. Thus the designers of the ENIAC incorporated additional "flip-flops" as buffers between the asynchronous signals and the remainder of the machine. The designers' reasoning as expressed in lectures and operating manuals was essentially as follows. The asynchronous signal, being random, could occur at a time when it caused passage of only a partial "clock" pulse thus making operation indefinite. By using this partial pulse, however, to set a flip-flop, and gating a later pulse through it, a "full" output pulse is obtained if the flip-flop sets and no output pulse if it remains unset. In the latter case, a full pulse will be obtained during the next cycle (since the asynchronous signal is of long duration) which will be certain to set the flip-flop. This scheme has, apparently, yielded equipment which functions quite satisfactorily. There is, however, a flaw in the above reasoning in the assumption that the flip-flop is definitely either set or not set when it gates the interrogating pulse. From our previous discussion, there must be a finite, although very small, probability that the gated pulse is also "partial." It seems obvious, nevertheless, that the method is effective in markedly reducing the probability of error.

This use of "trigger" circuits between an asynchronous signal and the remainder of the machine is, to the author's knowledge, resorted to by all computer designers. The purpose of this paper is to point out that the effect is to reduce but not eliminate probability of error, and that material im-

provement is possible by consideration of the various factors involved in such an arrangement. It is easy to see that the region of indefiniteness is decreased by "squaring" up of the signal and interrogating pulses, by speeding up of the "flipping" of the trigger circuit and by increasing the time interval between the pulse gated by the signal and acting upon the trigger and the pulse gated by the trigger and applied to the balance of the machine. The "squaring" of the signal and pulses reduces the probability of error in a smaller proportion than the decrease in rise or fall times since duration is a factor as well as slope. Moreover, the rise time cannot be readily increased indefinitely because of the usual circuit limitations. The improvement that can be achieved by careful attention to the shape of the pulse is thus quite limited. As concerns time of response of triggers and time interval between interrogating pulses, the probability of error will, in most cases, be an exponential function of the ratio of the two times. Decrease of this ratio by increasing the rate of response of the trigger or by increasing the time interval between gating and reading the trigger is therefore extremely effective in reducing probability of error.

We will conclude by indicating, for a particular configuration, how probability of error may be evaluated, at least as concerns order of magnitude. For this purpose, we assume perfectly "square" pulses and signal and a trigger whose output builds up exponentially to a steady state proportional to the input, and which is provided with positive "feed-back" for outputs in excess of both the input and a fixed noise suppression voltage. Such a trigger is approximated by a linear amplifier with ideal diode gating. If we take the input as E_1, and the trigger output as E, we have $E = AE_1(1 - e^{-t/k})$, where A is the steady state amplification, t is the duration of the input and k is the time constant of the circuit. If E ever reaches the value E_1, the feed-back replaces the signal and the trigger is fully set. If E does not reach the value E_0 which is the noise suppression voltage, no feed-back takes place when the input signal ends and the trigger is fully reset. We are interested in the intermediate condition where, at time t when the input ends, $E_1 + E_0 > E > E_0$. In this case, the feed-back becomes effective at time t and we get the equation

$$k\frac{dE}{dT} + E = A(E - E_0)$$

whose solution is

$$E = \left[AE_1(1 - e^{-t/k}) - \frac{AE_0}{A - 1}\right]e^{(A-1)T/k} + \frac{AE_0}{A - 1}$$

for the period following cessation of the first interrogating pulse, where T is the time interval during this period. Since $A > 1$ for satisfactory trigger operation, this solution represents an exponential build-up. If E falls between certain limits, say E_2 and E_3, at the time the trigger is examined, error results. The probability of error is thus the probability that t is such as to make

$$E_2 < \left[AE_1(1 - e^{-t/k}) - \frac{AE_0}{A - 1}\right]e^{(A-1)T/k} + \frac{AE_0}{A - 1} < E_3$$

so that $t_1 < t < t_2$, where

$$t_1 = -k \ln (S - E_2/AE_1) + (A - 1)T$$

and

$$t_2 = -k \ln (S - E_3/AE_1) + (A - 1)T,$$

where

$$S = \left[1 - \frac{E_0}{(A - 1)E_1}\right] e^{(A-1)T/k} + \frac{E_0}{(A - 1)E_1}.$$

If Q is the repetition period for the interrogating pulses, this probability is given by

$$P = \frac{(t_2 - t_1)}{Q}$$
$$= \frac{k}{Q} \ln \frac{S - E_2/AE_1}{S - E_3/AE_1}.$$

In view of the natural choice of T large compared to k and E_1 large compared to E_0, S is given approximately by $e^{(A-1)T/k}$ and P is approximated as

$$P \doteq \frac{k}{Q} \ln \left[1 + \frac{(E_3 - E_2)}{ASE_1}\right]$$
$$\doteq \frac{k}{Q} \frac{(E_3 - E_2)}{AE_1} e^{-(A-1)T/k},$$

which summarizes the important factors. If, in a particular case, we have $(E_3 - E_2)/E_1 = 0.1$; $A = 2$; $k = 1$ microsecond; $T = 10$ microseconds and $Q = 25$ microseconds, we find

$$P = \frac{1}{500} e^{-10} \doteq 10^{-7},$$

which is too high for comfort. However, increase of T or decrease of k by a factor of 2 makes

$$P = \frac{1}{500} e^{-20} \doteq 4 \times 10^{-12},$$

which may be acceptable. Similar evaluation of effectiveness may be made for other types of trigger circuits.

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BIBLIOGRAPHY Z-XXI

1. ANON., "Analog-digital converter," *Rev. Sci. Instr.*, v. 22, Oct. 1951, p. 800.

   Expository article.

2. ANON., "Digital computers," *The Electrician*, v. 148, March 14, 1952, p. 818.

   This is an expository article which very briefly describes certain features of the digital computer at Manchester University, England. This machine

is one of a series of machines which Ferranti, Ltd., is producing. The second one of this series will be located in Toronto, Canada, at the university.

EDITH T. NORRIS

NBSMDL

3. R. C. M. BARNES, E. H. COOKE-YARBOROUGH & D. G. A. THOMAS, "An electronic digital computor using cold cathode counting tubes for storage," *Electronic Engineering*, v. 23, Aug. 1951, p. 286–291, and Sept. 1951, p. 341–343.

In this sequence controlled computor, whose numbers contain eight decimal digits and a sign, speed was sacrificed to achieve small size and simplicity by using relays for switching and "Dekatron" 10-position cold cathode discharge tubes for up to 90 words of storage. Numbers and the instructions in sequence are read from perforated tape by up to eight tape readers and results are printed or punched by up to eight printers or perforators. The computor contains about 380 relays, 18 Dekatrons, 80 thermionic tubes, and 40 cold cathode triodes, plus 28 relays and 90 Dekatrons per 10 words of storage. It occupies three 7 ft. relay racks, together with an additional rack per 49 words of storage, one smaller rack for power supplies, and a table for tape readers and printers. Total power consumption is about 1 kilowatt.

The computor operates in parallel mode, and for transfer within the machine a decimal digit is represented by a train of 0 to 9 pulses in its appropriate channel while on the tapes a 2 out of 5 code is used. Round off is performed by adding 1 or 0 at random in the 8th place. Negative numbers are represented by 9's complements with end around carry correction.

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4. H. J. GEISLER, "R. F. bursts actuate gas tube switch," *Electronics*, v. 25, Feb. 1952, p. 104–105.

The article gives a brief description of a gas tube gate, requiring radio-frequency excitation of one of its inputs. The r-f energy is coupled to the tube through the d-c electrodes and a conducting band around the tube envelope. Cold open-circuit resistance is several megohms, and the capacitance between electrodes and also between the band and the electrodes is less than 1 $\mu\mu$f.

The characteristics of the IBM-36 developmental gas tube switch are given, and the application of the tube in computer storage and accumulator read-in circuits is discussed.

E. W. C.

NBSMDL

5. J. A. GOETZ & A. W. BROOKE, "Electron tube experience in computing equipment," *Electrical Engineering*, v. 70, Feb. 1952, p. 154–157.

The IBM Corporation has 2,500,000 electron tube sockets which are used in commercial computation equipment. Two years ago the IBM Tube Laboratories established a defective tube analysis program. This paper

describes the methods used for preventive maintenance testing of tubes before installation, and for analyses of causes of failure or rejection for several tube types used in number by IBM. It shows how an appreciable gain in machine reliability has been accomplished since this program has been in effect. The analysis of common causes of tube failures and vacuum tube life expectancy and survival data is of special interest to the computer design and maintenance engineer.

P. D. Shupe

NBSCML

6. W. E. Mutter, "Improved cathode-ray tube for application in Williams' memory system," *Electrical Engineering*, v. 71, Apr. 1952, p. 352–356.

This article describes the IBM-79 (RTMA 3VP1), a cathode-ray tube designed specifically for use in a Williams' electrostatic storage system. Attempts were made to reduce spot size, to improve "spill" characteristics and to reduce deflection defocusing and noise pickup. As in all tubes, the tube was designed to effect a compromise of conflicting requirements. The three-inch size represents a compromise between "bits" per tube and "bits" per unit volume.

Compared with the 3KP1, the best commercially available tube of the same size, there is a decided improvement as shown by the following: 1) spot separation for a given "spill" number is about 60 percent of that of the 3KP1 in a two-dot test; 2) spot size is about 76 percent of that of the 3KP1; and 3) deflection defocusing (from the illustrations) is about 90 percent of that of the 3KP1 at one inch from the center of the tube. These improvements were obtained by reducing the magnification ratio, masking down the beam, and shaping the deflection plates. In addition, the capacities of the deflection plates, especially to the grid, were balanced to reduce deflection caused by beam pulsing. Extreme cleanliness in assembly and processing and "sparking" were used to produce a larger percentage of blemish-free tubes. Noise pickup may be reduced by an external silver coating, grounded near the output end. The smaller spot size gives a reduced, but still ample, output signal. No attempt was made to reduce the "gentle rain" of secondary electrons on the storage surface.

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NBSCML

7. T. J. Rey, "On the background of pulse-coded computers, part 1," *Electronic Engineering*, v. 24, Jan. 1952, p. 28–32; part 2, *Electronic Engineering*, v. 24, Feb. 1952, p. 66–69.

Part 1 develops the background of the present day digital computer. Then the functions of and relations between the Input-Output, Control, Memory and Arithmetic units is explained, and the distinction is made between serial and parallel operations. The remainder and major portion of part 1 is concerned with the representation of numbers within a digital machine based on the binary number system and the congruence relationship of number theory.

Part 2 describes how the basic building blocks of digital computers may be represented by Boolean algebra. There are several examples of the design

of such sub-units as an adder from a truth table to a logical circuit. The article is concluded by a comparison between digital and analog systems.

WILLIAM A. NOTZ

NBSCML

8. JOHN J. WILD, "High-speed printer for computers and communications," *Electronics*, v. 25, May 1952, p. 116–120.

A novel printer, which has produced, under experimental operation, up to 900 eighty-character lines a minute without serious degradation of impression, has been developed by the Potter Instrument Company. The device consists of a "Flying Typewriter" and electronic control, the former being a rotating-type-wheel typewriter and the latter essentially a flexible electronic counter with certain special features incorporated to trigger 80 solenoid-controlled hammers facing the periphery of the type wheel. The time when a hammer strikes the type wheel through ribbon and paper determines the character printed in the corresponding column.

The control system automatically distributes the printing of the characters in the proper order during a revolution of the type wheel, although the order of printing is not necessarily in positional sequence around the wheel.

The hammers are resilient to cause them to bounce back swiftly after striking the type. Time of contact of hammer and type face is under 0.1 millisecond, and the time of operation of the hammer is about 2.5 milliseconds. The pulses operating the hammers are timed with a lead of 2.5 milliseconds to cause the striking of the type character in the center. Even at extremely high rates of operation no appreciable blur of print occurs. The printing is equivalent, in clearness, to that of a good typewriter.

The time of operation of each of the 80 hammers is controlled by information set into a trigger tube storage called a PASS unit (printer actuator serial storage unit). A motor-driven notched disc and phototube arrangement provide the necessary synchronizing pulses. The PASS unit has 80 columns of 6 binary-digit storage, with shifting circuits for loading and special gating circuits for driving the printer. PASS may be loaded a column at a time, with a 6 binary-digit code, or it may be loaded a pulse at a time. Each of its 80 columns can be used as a scale of 10 or 64 counter. In fact, the PASS unit is essentially an accumulator and shift register and is capable of use by an automatic computer as a part of its arithmetic unit, when printing is not being performed.

The loading of the PASS unit with the information to be printed may be thought of as the pre-setting of the 80 counters in the unit. The pulses generated by the photo-electric disc are fed individually to step the counters, with time lag between columns to prevent the call for printing of the same character in more than one column at the same time. When each of the various counter columns in the PASS overflows, an output pulse is generated which causes the firing of a thyraton and the consequent energizing of the solenoid-controlled hammer. The characters around the rotating-type wheel and the photo-electric disc are so placed that when the hammer strikes, the correct type character is opposite it.

The new Potter Instrument Company printer was designed to operate at high speed, and also to be capable of accepting data for printing from punched cards, tapes, or directly from automatic digital computers. Presetting of selected columns in the control storage inhibits printing in the corresponding columns of a line; this feature facilitates the arrangement of the output format. This feature, together with the high printing speed, if the printer is reliable, should make the unit a useful addition to high-speed computer accessories.

E. W. C.

NBSMDL

News

Institute for Advanced Study.—A high-speed electronic digital computing machine has been completed and put into operation at the Institute for Advanced Study in Princeton, New Jersey. The machine is designed to perform very high-speed calculations in pure and applied mathematics and in mathematical physics.

Prior to this public announcement the machine had successfully completed a number of quite extensive and important problems. These problems include among others the following:

1) A large number-theoretical problem to test a conjecture, which has never been established, of the famous 19th century mathematician, E. E. Kummer. This calculation required the instrument to perform about 20,000,000 multiplications and took six hours of continuous computing.

2) Two considerably shorter astro-physical problems, each requiring the solution of three simultaneous differential equations.

3) Solutions of a number of cubic diophantine equations.

4) Several twelve-hour meteorological predications covering the continental United States, each amounting to about 800,000 multiplications and requiring about one hour of continuous computing time. This last work is only the first step in an extensive research program in theoretical meteorology being carried out at the Institute for Advanced Study.

This machine took six years to design, develop and construct by a staff under the direction of Prof. JOHN VON NEUMANN at the Institute for Advanced Study. The initial sponsorship of the project came from the Research and Development Service of the Ordnance Corps, U. S. Army. It has continued under the joint sponsorship of that agency, together with the Office of Naval Research, U. S. Navy; the U. S. Air Force; and the U. S. Atomic Energy Commission. Throughout its history, the project had the support and encouragement of the Institute for Advanced Study. In addition, the Office of Naval Research, since 1946, and since 1951 in cooperation with the Geophysics Research Division, Air Force Cambridge Research Center, Cambridge, Mass., has sponsored the work in dynamic meteorology. The Office of Naval Research also supported a complementary research program in numerical analysis.

The machine has been the prototype for various subsequent machine developments, including three for the AEC and one recently completed by the University of Illinois for the U. S. Army Ordnance Corps, Ballistic Research Laboratories at the Aberdeen Proving Ground.

The engineering design was due to JULIAN BIGELOW, its execution to Julian Bigelow and JAMES POMERENE, assisted by GERALD ESTRIN, HEWITT CRANE, RICHARD MELVILLE, NORMAN EMSLIE, and EPHRAIM FREI, as well as by GORDON KENT, PETER PANAGOS, and others. The mathematical and logical design was due to John von Neumann and HERMAN H. GOLDSTINE. The work on meteorology is under the joint direction of JULE CHARNEY and John von Neumann.

Data can be introduced into the machine in decimal or in binary form, but the instrument proper carries out the calculation in the binary number system since the use of this number system is more convenient electronically. Each number handled by the machine consists of a sign and 39 binary digits which is the equivalent of a decimal number with a sign and approximately 12 decimal places. The machine produces 2,000 multiplications per second, 1,200 divisions per second or 100,000 additions per second. For a machine of this degree of precision it is the fastest one now operating. The machine consists of four principal organs: namely, an arithmetic organ which carries out the processes indicated above, a control organ which executes the instructions given the machine, a memory organ in which both the numerical data of the problem and the instructions which characterize the problem are stored and, lastly, an input-output organ which intervenes between the human operator and the machine itself. The memory is a system of 40 cathode ray tubes, based on an invention of F. C. Williams in Manchester, England. The machine can get access to the memory organ in 25 microseconds. This organ is capable of storing 1,024 numbers each of 40 binary digits.

It is unusually small in physical size, being approximately $2 \times 8 \times 8$ feet, and its total power requirements including ventilators, etc., are about 15 kilowatts. It contains about 2,340 vacuum tubes, almost all of which are double triodes.

Reeves Instrument Corporation.—Project Cyclone Symposium II on Simulation and Computing Techniques was held in New York City, April 29–May 2, 1952, under the sponsorship of the Reeves Instrument Corporation with the approval of the U. S. Navy Special Devices Center. The program consisted of three sessions, under the chairmanship of Rawley D. McCoy, Reeves Instrument Corporation.

Session I	
Analogue Computer Techniques and Applications	Wednesday, April 30, Morning:
REAC solution of problems in structural dynamics	C. W. Brenner, Mass. Inst. of Technology
The use of an analogue computer and feedback theory for the solution of structural problems in the static case	G. Martin and L. M. Legatski, University of Michigan
Application of the Electronic Differential Analyzer to eigenvalue problems	G. M. Corcos, R. M. Howe, L. L. Rauch and J. R. Sellars, University of Michigan
Some REAC techniques employed at the David Taylor Model Basin	L. Pode, David Taylor Model Basin
	Afternoon:
Simulation studies of a relay servomechanism	N. P. Tomlinson, Goodyear Aircraft Corp.
Use of the REAC as a curve fitting device	C. H. Murphy, Ballistic Research Laboratories, Aberdeen Proving Ground
Precision in high-speed electronic differential analyzers	H. Bell, Jr., and V. C. Rideout, University of Wisconsin
On an application of the use of analogue computers to methods of statistical analysis	J. H. Laning, Jr., and R. H. Battin, Massachusetts Inst. of Technology
Solution of linear differential equations with time varying coefficients by the Electronic Differential Analyzer	C. E. Howe, R. M. Howe and L. L. Rauch, University of Michigan
Session II	
General Papers	Thursday, May 1, Morning:
JAINCOMP computers and their application to simulation problems	D. H. Jacobs, The Jacobs Instrument Co.

The Decimal Digital Differential Analyzer CRC 105 as a tool for simulation and checking analogue computer solutions	E. Weiss, Computer Research Corp.
Problems encountered in the operation of the MIT Flight Simulator	W. W. Seifert and H. Jacobs, Jr., Massachusetts Institute of Technology
Automatic REAC operation for statistical studies	R. R. Bennett and A. S. Fulton, Hughes Aircraft Co.
	Afternoon:
Mathematical error analysis for continuous computers	F. J. Murray, Columbia University
Checking analogue computer solutions	W. F. Richmond, Jr., and B. D. Loveman, The Glenn L. Martin Co.
Analogue computation of blade designs	D. B. Breedon, M. M. Matthews and E. L. Harder, Westinghouse Electric Corporation
Solution of partial integral-differential equations of electron dynamics using analogue computers with storage devices	C. C. Wang, Sperry Gyroscope Co.
Session III	
Computer Components	Friday, May 2, Morning:
Modification of REAC	J. W. Follin, Jr., G. F. Emch and F. M. Walters, Applied Physics Laboratory, Johns Hopkins University
REAC servo response	A. H. Miller, University of Minnesota
Applications of differential relays to solution of REAC problems	L. M. Warshawsky and W. Braun, Wright Air Development Center
The design and test of a Linear Wiener Filter	G. Nestor, Avion Instrument Corp.
	Afternoon:
The role of diodes in an Electronic Differential Analyzer	C. D. Morrill and R. V. Baum, Goodyear Aircraft Corp.
A high accuracy time division multiplier	E. A. Goldberg, Radio Corporation of America
An AM–FM Electronic Analogue Multiplier	W. A. McCool, Naval Research Laboratory
Discussion Period on Electronic Multiplication	

UNIVAC Acceptance Tests.—On April 22, 1952, the third UNIVAC system, this one constructed for the U. S. Army Map Service by the Eckert-Mauchly division of Remington-Rand, Inc., under NBS contract, passed the final test for its acceptance. This was the same magnetic tape reading and writing test given the second UNIVAC (see *MTAC*, v. 6, Apr. 1952, p. 119). It required reading 142 million decimal digits and writing 85 million; however, in this test the central computer controlled 10 tape systems. Error detecting circuits stopped the computer in 8 out of 25 fifteen-minute test units; however, there were no undetected errors, and the operator in every case corrected the trouble without assistance from an engineer or maintenance technician. That this rate of stoppages, barely low enough for acceptance, was not achieved until the seventh attempt at the test indicates that a considerable improvement in reliability of tape reading would be desirable. However, this should not be used for unfavorable comparison with any other machine. To this writer's knowledge no other machine approaches the UNIVAC's speed in reading and writing on magnetic tapes, much less has any other computing system been subjected to so rigorous a test.

OTHER AIDS TO COMPUTATION

Bibliography Z-XXI

9. Anon., "The Radar Range Calculator," *Bell Lab. Record*, v. 29, 1951, p. 312–313.

A circular slide rule calculator for computing the function of seven parameters which describes the effective range of a radar system. This function is a product of fractional powers of the parameters.

F. J. M.

10. A. A. Currie, "The general purpose analog computer," *Bell Lab. Record*, v. 29, March 1951, p. 101–108.

This article describes an analog computer using a basic three stage feedback amplifier for integration, differentiating and forming linear combinations and with servo driven potentiometers for multiplication and finding functions like sin x. The computer contains 30 adders, 1 differentiator, 12 integrators, 9 servos which can drive 45 linear potentiometers and 15 function potentiometers, with 41 additional potentiometers which can be set by hand. There are six output recorders, two of which may be modified to be used as function input tables from graphs. The machine is set up by means of a three panel patch bay.

F. J. M.

11. L. L. Grandi & D. Lebell, "Analogue computers solve complex problems," *Radio and Television News*, Radio Electronic Eng. ed., v. 46, No. 5, Nov. 1951, p. 70–71, 138–139.

The Engineering Department of the Univ. of Calif. at Los Angeles has four continuous computers, an electrical differential analyzer, a mechanical differential analyzer, a network analyzer and a thermal analyzer. This article describes the problems for which each may be used and contains eight photographs including a color photograph on the cover.

F. J. M.

12. W. G. James, *Logarithms in Instrumentation*. United States Atomic Energy Commission ORNL-413, Oak Ridge National Laboratory, Oak Ridge, Tenn., 54 p.

The author surveys the methods used for taking logarithms in instruments and indicates the accuracy and ranges available from the different methods. The logarithmic devices described include diodes with negative anodes, contact rectifiers, variable μ tubes and potentiometers either wound with a special taper or with shunts between taps.

F. J. M.

13. L. H. Johnson, *Nomography and Empirical Equations*. New York, Wiley; London, Chapman & Hall, 1952, ix + 150 p., 22.9 × 14.3 cm. Price $3.75.

This book is an elementary text with the minimum of theory, and judged as such it is excellent. In particular the author considers questions of arranging nomograms for maximum accuracy. He also has a number of

good points in curve fitting, though his treatment of least square fitting could be improved a bit.

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14. E. LAKATOS, "Problem solving with the analog computer," *Bell Lab. Record*, v. 29, March 1951, p. 109–114.

A process is given for translating a differential equation into a form to which the differential analyzer is applicable. Two examples are briefly discussed.

F. J. M.

15. Y. W. LEE & J. B. WIESNER, "Correlation functions and communication applications," *Electronics*, v. 23, June 1950, p. 86–92.

This article describes, largely in quantitative terms, some of the techniques and applications arising out of recent developments in the theory of communication, based on the statistical concept of information. The meanings of autocorrelation functions and cross-correlation functions are discussed for stationary random processes, and these functions are applied to periodic processes. An electronic correlator, built at MIT, for obtaining and recording the autocorrelation curve of a random function is briefly described. The correlator obtains two sets of sampled values of the random function at regular intervals and forms pulses whose height and width are proportional to the two sets of sampled values, respectively. The autocorrelation function is found (approximately) by integrating the pulse areas.

The problem of detection of periodic signals is discussed, and experimental results obtained with the correlator, showing a 30 db. gain for a signal which is 15 db. below the noise level, are presented. Additional applications of the correlator to the determination of the impulse response of networks and the measurement of crosstalk in multichannel communication systems are indicated briefly. Finally, the importance of correlation functions in Wiener's statistical prediction and filter theory is discussed. More quantitative information on many of the topics included in this article may be found in another paper by the same authors.[1] Details concerning the correlator are given in a technical report.[2]

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[1] Y. W. LEE, T. P. CHEATHAM, JR., & J. B. WIESNER, "Application of correlation analysis to the detection of periodic signals in noise," I. R. E., *Proc.*, v. 38, 1950, p. 1165–1171.

[2] T. P. CHEATHAM, "Experimental determination of correlation functions and their application to the statistical theory of communications," *Technical Report* No. 122, Research Laboratory of Electronics, MIT.

16. A. B. MACNEE, "Some limitations on the accuracy of electronic differential analyzers," I. R. E., *Proc.*, v. 40, 1952, p. 303–308.

The author considers the errors that arise in solving a linear differential equation

$$\sum_{n=0}^{m} A_n y^{(n)} = F(t) \tag{1}$$

with constant coefficients by means of an electronic analogue computer. Only adders and integrators are necessary. The steady state response of the simplest physically realizable adder is of the form

$$E_{out} = K(E_1 + E_2 + \cdots + E_n)/(1 + j\omega t_1).$$

Similarly, for an integrator

$$E_{out}/E_{in} = [1/j\omega k][1/(1 + 1/j\omega\mu k)][1/(1 + j\omega T)]$$

where μ is the gain of the base amplifier and the last factor is due to the imperfect high frequency response of a physical integrator. Using these approximations, the author derives simple approximate expressions for the difference e_n, $n = 1, \cdots, m$ of the characteristic roots s_n (assumed distinct) of (1) and the characteristic roots s_n' of the differential analyzer. The $m + 1$ additional roots of the machine solution [introduced by the "$1/(1 + j\omega T)$" factors which raise the order of (1)] are, under physically reasonable assumptions, widely separated from the s_n roots and have negative real parts. Numerical examples are included. The case in which (1) has multiple roots is briefly touched upon; the phenomena which arise when the Jordan normal form is not diagonal are not mentioned.

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17. R. E. Scott, *An analog device for solving the approximation problem of network synthesis. Technical Report* No. 137, June 8, 1950, *Research Laboratory of Electronics*, Massachusetts Institute of Technology, Cambridge, Mass.

Mathematically the problem treated here is that in which one endeavors to find a rational function which is a good approximation on the imaginary axis to a prescribed complex valued function. The desired function is the "network response" function from which a network can be synthesized with a given frequency response corresponding to the prescribed values along the imaginary axis. The rational function must satisfy a number of conditions, for instance, the numerator and denominator must be expressible as real polynomials and the roots of the denominator must all be in the left half of the complex plane. It is possible to represent the logarithm of a rational function with a prescribed set of zeros and poles as the potential function of a planar flow of electric current with corresponding sources and sinks.

The device described in this report has as its conducting plane a sheet of "Teledeltos" paper. Current sources are provided and also probes which are stuck into the paper to represent the sources and sinks. The values of the potential function along the imaginary axis are sampled from fixed contacts by a commutator and displayed on an oscilloscope. The movable probes are adjusted until the desired behavior is obtained. One major problem in this process is due to the finite extent of the conductor. A number of solutions for this have been proposed.[1] In this device, a logarithmic transformation is made which with certain symmetry considerations permits one to substitute a strip instead of the infinite plane. The strip is termi-

nated at each end at some distance from the computing portion to represent zero and infinity. To obtain the equivalent of phase the fixed contacts along the imaginary axis are doubled so that potential slope can be indicated. The report contains considerable information on both the construction and use of this device and an analysis of the effect of individual errors. A photograph shows the entire machine mounted on a laboratory cart.

F. J. M.

[1] Cf. A. R. Boothroyd, E. C. Cherry & R. Makar, Inst. Elect. Eng., *Proc.*, v. 96, 1949, p. 163–177; *MTAC*, v. 5, p. 49–50.

18. J. R. Shah & L. Jacobs, "Investigation of field distributions in symmetrical electron lens," *Jn. Appl. Physics*, v. 22, 1951, p. 1236–1241.

The potential field for the electron lens is obtained by the use of an electrolytic tank. It was determined experimentally that an impedance of 30 megohms for the probe was necessary. With these methods, the crossing of the nodes at the central symmetry point was accurate to within about a quarter of a degree. The field obtained from the electrolytic tank was compared with that obtained by a relaxation method. The maximum difference is about 1.5%.

F. J. M.

19. B. A. Sokoloff, "Principe et réalisation d'une machine mathématique dite 'Operateur Mathématique Electronique' OME," *Annales des Télécommunications*, v. 5, no. 4, April 1950, p. 143–159.

The mathematical machine described is an electronic differential analyzer with the customary feedback amplifier integration and addition. The output indicator is an oscilloscope and camera combination. Multiplication is by servo driven potentiometers. A discussion of stability is given for the constant coefficient case of linear differential equations, including the use of the determinantal conditions for positive definiteness.

F. J. M.

20. G. J. Tauxe & R. L. Stoker, "Analytical studies in the suppression of wood fires," A. S. M. E., *Trans.*, v. 73, 1951, p. 1005–1020.

Different methods of suppression of wood fires are analyzed thermally using electrical analogy methods. The latter effect the solution of the heat equation by means of an R-C network. The method has been previously described.[1]

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[1] V. Paschkis & H. D. Baker, "A method for determining unsteady-state heat transfer by means of an electrical analogy," A. S. M. E., *Trans.*, v. 64, 1942, p. 105–110.

NOTES

139. The RAND Collection of Illustrative Approximations.—In recent years, the Numerical Analysis Department at RAND has been preparing loose leaf sheets that contain interesting and useful approxima-

tions to a number of the higher transcendental functions. The data from a sample sheet are displayed at the end of this note together with a photographic reproduction of the error curve appearing on this sheet. Some sixty odd sheets in this series have now been prepared, and up to date collections of these sheets have been distributed to some three hundred people working in the field of numerical analysis throughout the United States and in a few foreign countries.

The approximations given in this series of loose leaf sheets are of both a practical and of an illustrative nature. In a high speed digital machine, these approximations may take the place of bulky tables or the place of awkward series developments. For the hand computer, an easily evaluated expression will often be of use when a required table is unavailable for consultation. Beyond this, however, the sheets will be useful for the insight they give the practical computer in the approximation of functions. Starting with approximations concerning the common logarithm, sheets for an ever increasing number of the useful transcendental functions have been prepared and distributed. Sheets of approximations in this series are already available concerning the logarithmic function, the exponential function, the inverse tangent, the sine function, the inverse sine, the Gamma function, the Gaussian error integral, the inverse Gaussian error integral, the complete elliptic integrals of first and second kind, the exponential integral and a number of other special functions.

As the approximations in this series are intended to be illustrative as well as practical, great care has been taken in the accurate leveling of the error curves. The sheets of approximations thus give interesting information concerning the location of roots—points at which function and approximation agree—and the location of extremals—points at which the deviation between function and approximation is locally a maximum in an absolute or relative sense. The accurate and carefully drawn error curves that appear on each sheet will do much to give the reader a feel for the nature of analytical approximation in a wide variety of typical cases of practical importance. In order to provide this important information, the coefficients in the approximations given have been recorded to perhaps two, three or four more decimals than would be required for practical considerations of utility. Should the coefficients given be of awkward size for use in a given computing machine, it is permissible to round the numbers quite severely. Of course the beauty and character of the error curve will be lost as a result of such mistreatment, but this is not a matter of practical concern in the use of the approximation.

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Approximations in Numerical Analysis

Function:

$$-\mathrm{Ei}\,(-X) = \int_X^\infty \frac{e^{-t}\,dt}{t}$$

Range:

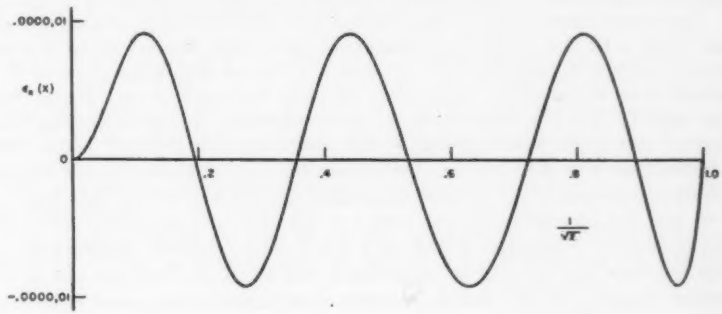
$$1 \leq X < \infty$$

Approximation:

$$-\mathrm{Ei}^*(-X) = \frac{e^{-X}}{X}\left\{\frac{a_0 + a_1X + a_2X^2 + X^3}{b_0 + b_1X + b_2X^2 + X^3}\right\}$$

$$\begin{aligned} a_0 &= .2372,9050 & b_0 &= 2.4766,3307 \\ a_1 &= 4.5307,9235 & b_1 &= 8.6660,1262 \\ a_2 &= 5.1266,9020 & b_2 &= 6.1265,2717 \end{aligned}$$

Error Curve (Approximation − Function)/(Function):



140. AN ALTERNATIVE "PREDICTOR-CORRECTOR" PROCESS.—The purpose of this note is to report the successful use, on a system of 3 non-linear ordinary differential equations of order 2 (with initial conditions), of a pair of formulas analogous to Milne's.[1]

The system of differential equations is of the form

$$\ddot{x}_i = f_i(\dot{x}_1, \dot{x}_2, \dot{x}_3, x_1, x_2, x_3, t), \quad (i = 1, 2, 3). \tag{1}$$

The formulas used for *prediction* are

$$\dot{x}_i^{(n+1)} = 5\dot{x}_i^{(n-1)} - 4\dot{x}_i^{(n)} + 2h[\ddot{x}_i^{(n-1)} + 2\ddot{x}_i^{(n)}], \quad (i = 1, 2, 3). \tag{2}$$

The *correction* formulas are

$$\dot{x}_i^{(n+1)} = \dot{x}_i^{(n)} + \frac{h}{12}[5\ddot{x}_i^{(n+1)} + 8\ddot{x}_i^{(n)} - \ddot{x}_i^{(n-1)}], \quad (i = 1, 2, 3) \tag{3}$$

and their companions

$$x_i^{(n+1)} = x_i^{(n)} + \frac{h}{12}[5\dot{x}_i^{(n+1)} + 8\dot{x}_i^{(n)} - \dot{x}_i^{(n-1)}], \quad (i = 1, 2, 3). \tag{4}$$

In the above, h denotes the integration step.

The process is as follows: Given two sets of "starting" values, $x_i^{(n-1)}$, $\dot{x}_i^{(n-1)}$, $\ddot{x}_i^{(n-1)}$ and $x_i^{(n)}$, $\dot{x}_i^{(n)}$, $\ddot{x}_i^{(n)}$ [the values at $t = t_{n-1}$ and $t_n = t_{n-1} + h$, resp.], the first approximations ${}_1\dot{x}_i^{(n+1)}$ are predicted by (2), then (4) are used to obtain ${}_1x_i^{(n+1)}$. These first approximations are substituted in (1) to obtain ${}_1\ddot{x}_i^{(n+1)}$, which are then used in (3) to obtain ${}_2\dot{x}_i^{(n+1)}$. The above steps may be repeated, to give ${}_3\dot{x}_i^{(n+1)}$, etc. If the ${}_j\dot{x}_i^{(n+1)}$ prove acceptable (see below), then ${}_jx_i^{(n+1)}$ are computed and accepted.

An estimate of the error in the second approximation to $\dot{x}_i^{(n+1)}$ is available from the error terms of (2) and (3), viz. $h^4 x_i^v(\xi)/6$ and $-h^4 x_i^v(\eta)/24$, respectively, where both ξ and η are on the interval $t_{n-1} < t < t_{n+1}$ and where x_i^v denotes the fifth derivative of x_i. If we let $D_{ij} = {}_{j+1}\dot{x}_i^{(n+1)} - {}_j\dot{x}_i^{(n+1)}$, then, as in Milne,[1] we find that the error in ${}_2\dot{x}_i^{(n+1)}$ is approximately $D_{i1}/5$. Thus, if all the $|D_{i1}/5|$ are insignificant, one iteration is probably sufficient, whereas if any $|D_{i1}/5|$ is intolerably large, a smaller h and/or more iterations are called for.

In the problem to which the process was applied (it was carried out on our IBM CPC), $h = 0.1$ was chosen initially. This proved to be a good choice, with 2 iterations, from $t = 0$ to $t = 4.0$, as shown by D_{i1} and D_{i2}, which were all listed for monitoring. In most instances, some $|D_{i1}/5|$ was too large ($> \frac{1}{2}\cdot 10^{-4}$), whereas no $|D_{i2}|$ exceeded 10^{-3}, thus indicating that further iterations would probably not affect the 3rd decimal place. The combination $h = 0.1$, and two iterations, was actually used to $t = 4.4$, at which place some $|D_{i2}|$ increased significantly; the h was then decreased to 0.05 and the process continued from $t = 4.0$ with two iterations (although the previously computed values at $t = 4.3$ were probably satisfactory, it was felt best, for the sake of smoothness and safety, to back up to $t = 4.0$ before changing to $h = 0.05$). The new combination gave good results as far as $t = 5.15$, when some $|D_{i2}|$ again became significant. Here it was decided to use $h = 0.05$ again, but increase to 4 iterations; this worked successfully as far as $t = 4.90$.

It may be of interest to note that the need for smaller h and/or more iterations arose because the f_i in (1) each would (apparently) have become infinite in the neighborhood of $t = 4.7$. The results were checked by differencing the (accepted) values of the x_i. In addition, there was an internal check available in the problem, viz. that $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = K$. Interestingly enough, this check failed approximately at the same time that the last computed $|D_{ij}|$ became intolerably large.

One of the advantages of this process is that it requires only two sets of "starting" values rather than four. This means that in case it becomes necessary to decrease h at some point of the process, only one additional "point" is needed. This added point can be calculated, e.g., by means of the modified Euler method,[2] as can the additional point needed initially. Another "advantage" (rather a nebulous one, actually, since it is based on esthetics rather than mathematics) is that one has a feeling of greater confidence in values based on others in their more immediate neighborhood (two starting values as opposed to four). Obviously, examples exist which would vitiate this belief.

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The preparation of this paper was sponsored (in part) by the Flight Research Laboratory, Wright Air Development Center, USAF.

[1] W. E. Milne, *Numerical Calculus*. Princeton, 1949, p. 135.
[2] J. B. Scarborough, *Numerical Mathematical Analysis*. Baltimore, 1950, p. 235.

141. A New Calculation of Euler's Constant.—In 1878 J. C. Adams published his notable paper[1] on the evaluation of Euler's constant. By two separate calculations he determined approximations which agreed to 263D.

Since that time no attempt seems to have been made to extend the approximation beyond that point, until a recent study of Adams' work induced me to undertake the task.

Following the procedure adopted by Adams, I used the Euler-Maclaurin summation formula applied to the harmonic series, which yields the equation

$$\gamma = \sum_{k=1}^{n} k^{-1} - \ln n - \frac{1}{2n} + \sum_{r=1}^{m} \frac{(-1)^{r+1}B_r}{2rn^{2r}} + R_m.$$

For fixed n, the minimum absolute value of R_m is nearly equal to the numerical value of the term of the asymptotic series given by taking $m = [n\pi - \frac{1}{4}]$.

For this calculation of γ it was decided to take $n = 1000$, principally because of the resultant ease of evaluating the sum of the asymptotic series to about 350D.

The evaluation of the sum of the first thousand terms of the harmonic series to 350D was accomplished by means of an artifice used by Adams. He replaced that sum by an equivalent, namely $-43 + \sum_{i=1}^{168} a_i p_i^{-k_i}$, where $p_i^{k_i} < 1000 < p_i^{k_i+1}$, $k_i \geq 1$, and a_i is a positive integer $< p_i^{k_i}$ determined uniquely by decomposing the terms of the partial sum of the harmonic series into partial fractions with prime-power denominators, adding all the component fractions involving the same prime p_i, and finally reducing the numerator modulo $p_i^{k_i}$. This part of the calculation also was facilitated by the use of an accurate manuscript table of the complete decimal periods of the reciprocals of all primes less than 1000, which I had previously calculated in order to check Gauss' table of circulating decimals.[2]

The only remaining datum required was ln 1000. This was deduced from the 330D approximation to ln 10 calculated by H. S. Uhler.[3]

Inasmuch as Uhler guaranteed his value of ln 10 to within 2 units in the 329th decimal place, and all other parts of this calculation were carefully checked, it is believed that the following approximation to Euler's constant is correct to at least 328D. It confirms the accuracy of Adams' approximation to 262D.

$\gamma =$ 0.57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079
84793 74508 57057 40029 92135 47861 46694 02960 43254 21519
05877 55352 67331 39925 40129 674(28)

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[1] Roy. Soc. London, *Proc.*, v. 27, 1878, p. 88–94.
References to earlier calculations are given in FMR, *Index*, p. 95.
[2] *MTAC*, v. 4, 1950, p. 222, 223.
[3] Nat. Acad. Sci., *Proc.*, v. 26, 1940, p. 205–212.

142. The Determination of a Large Prime. [Editorial Note: The primality of $(2^{148}+1)/17$ as established by A. Ferrier was very briefly announced in *MTAC*, v. 6, p. 61. Since this is probably the last "largest" prime to be identified by hand computing methods (primes like $2^{1279}-1$ would require more than a century of desk calculator work to establish by any known method), it may be of interest to give the following details quoted from Ferrier's letter of July 14, 1951.]

I have established that

$$N = (2^{148}+1)/17 = 2098\ 89366\ 57440\ 58648\ 61512\ 64256\ 61022\ 25938\ 63921$$

is a prime and is in fact the largest prime known. The method used is the following.

I first found that N divides $3^{17N}-3^{17}$. In fact for $n = 2^{144}$ we have

$$\begin{aligned} 3^n &\equiv -566\ 68397\ 79443\ 57425\ 64352 \\ &\equiv U \pmod{N} \end{aligned}$$

and

$$U^2 \equiv 3^{2n} \equiv -9 \pmod{N}.$$

Hence

$$3\cdot(3^n)^{16} = 3^{17N} \equiv 3\cdot(-9)^8 \equiv 3^{17} \pmod{N}.$$

That is, N divides $3^{17N}-3^{17}$. Next we see that

$$N-1 = (2^{148}-2^4)/17 = 2^4(2^{72}-1)(2^{72}+1)/17.$$

The largest prime factor of $N-1$ is that dividing $2^{72}+1$ and is

$$p = 48\ 78248\ 87233.$$

Writing $N-1 = pm$ we find that $3^{17m}-1$ is prime to N. Applying the theorem quoted and corrected in *MTAC*, v. 3, p. 497, and v. 5, p. 259, respectively, it follows that every prime factor of N is of the form $px+1$ as well as $296x+1$, that is of the combined form

$$14439\ 61666\ 20968x + 1 = qx + 1.$$

For $q = 1(1)11$, $qx+1$ is divisible by small primes. Hence every prime factor of N exceeds $12q$.

Writing $N = A^2 - B^2$ we have

$$2A < 12q + N/(12q) < 1.3\cdot 10^{28}.$$

However

$$\begin{aligned} 2A &\equiv N + 1 \pmod{q^2} \\ &\equiv 1885\ 97808\ 71263\ 54966\ 31614\ 94450 \pmod{q^2} \end{aligned}$$

so that $2A > 1.8\cdot 10^{28}$.

Thus A does not exist and N is a prime.

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QUERIES—REPLIES

50. A DEFINITE INTEGRAL (Q 41, v. 6, p. 125).
In the definite integral

$$I = \int_0^\infty u^{-1} \exp(-zu - u^{-2})du$$

put $z = 2x^3$, $v = xu$ so that

$$I = \int_0^\infty v^{-1} \exp[-x^2(2v + v^{-2})]dv.$$

In this form the integral can be evaluated asymptotically by LAPLACE's method.[1] $f(v) = 2v + v^{-2}$ has its maximum at $v = 1$, and for large x the integral is approximated by

$$\left[\frac{2\pi}{x^2 f''(1)}\right]^{\frac{1}{2}} \exp[-x^2 f(1)]$$

or

$$(\pi/3)^{\frac{1}{2}} x^{-1} e^{-3x^2}.$$

This approximation may be used to start an asymptotic expansion: successive terms may be computed from the differential equation stated in the query.

A. E.

[1] See D. V. WIDDER, *The Laplace Transform*. Princeton, 1946, p. 277–280.

EDITORIAL NOTE: DR. J. ERNEST WILKINS also obtained the dominant term of the asymptotic expansion of this integral for large values of z.

CORRIGENDA

v. 3, p. 355, l. −12, *for* 10^{33} *read* 10^{35}.
v. 6, p. 20, l. −5, *for* $n < 50$ *read* $n \leq 50$.
v. 6, p. 25, l. 14, *for* 1.2(.5) *read* 1.2(.05).
v. 6, p. 25, l. 14, *for* 2–3D *read* 3–4D.
v. 6, p. 25, l. −20, *for* R52 *read* R53.
v. 6, p. 32, l. −7, *for* JOHNSON *read* JOHNSTON.
v. 6, p. 34, l. 7, *for* 1950 *read* 1949.
v. 6, p. 34, l. 9, *for* 21 *read* 206.
v. 6, p. 55, l. −4, *for* y_i *read* y_1.
v. 6, p. 58, l. 18, *for* 33 *read* 5.
v. 6, p. 58, l. 20, *for* 35 *read* 5.
v. 6, p. 61, l. −8, *for* 116 *read* 118.
v. 6, p. 82, l. 6, *for* TABLIS͡TY *read* TABLIT͡SY.
v. 6, p. 86, l. −17, *for* $\rho = 0.6$ *read* $\rho = 0.0$.
v. 6, p. 104, l. −11, *for* 7956 *read* 7556.

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CLASSIFICATION OF TABLES

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